**CHAPTER 2:**

**Mathematics for Microeconomics**

The problems in this chapter are primarily mathematical. They are intended to give students some practice with the concepts introduced in Chapter 2, but the problems in themselves offer few economic insights. Consequently, no commentary is provided. Results from some of the analytical problems are used in later chapters, however, and in those cases the student will be directed to here.

**Solutions**

**2.1** 

 a.  

 b. $U\_{x}=8, U\_{y}=12$

c. 

d. 

e. 

f. 

g. The  contour line is an ellipse centered at the origin. The slope of the line at any point is given by 

**2.2** a. Profits are given by  The maximum value is

 found by setting the derivative equal to 0:

 

 implies and 

 b. Since this is a global maximum.

 c.   So obeys

 

**2.3** First, use the substitution method. Substituting  yields  Taking the first-order condition,  and solving yields  and  Since this is a local and global maximum.

Next, use the Lagrange method. The Lagrangian is  The first-order conditions are

 

Solving simultaneously,  Using the constraint gives  and 

**2.4** Setting up the Lagrangian,  The first-order conditions are

 

So  Using the constraint  gives  and Note that the solution is the same here as in Problem 2.3, but here the value for the Lagrangian multiplier is the reciprocal of the value in Problem 2.3.

**2.5** a. The height of the ball is given by The value of  for

which height is maximized is found by taking the first-order condition:  implying 

 b. Substituting for 

 

 Hence,

 

c. Differentiation of the original function at its optimal value yields

 

 Because the optimal value of  depends on 

 

as was also shown in part c.

d. If  Maximum height is  If  maximum height is  a reduction of 0.08. This could have been predicted from the envelope theorem, since

 

**2.6** a. This is the volume of a rectangular solid made from a piece of metal

 which is  by  with the defined corner squares removed.

 b. The first order condition for maximum volume is given by

 

 Applying the quadratic formula to this expression yields

 

 The second value given by the quadratic  is obviously extraneous.

 c. If 

 

 So volume increases without limit.

 d. This would require a solution using the Lagrangian method. The optimal solution requires solving three non-linear simultaneous equations, a task not undertaken here. But it seems clear that the solution would involve a different relationship between  and  than in parts a–c.

**2.7** a. Set up the Lagrangian: The first-order

conditions are

 

 Hence,  With the optimal solution is 

 b. With  solving the first order conditions yields  and 

 c. If all variables must be non-negative, it is clear that any positive value

for  reduces  Hence, the optimal solution isand

d. If optimal solution is   Because 

provides a diminishing marginal increment to  whereas  does not, all optimal solutions require that, once  reaches 5, any extra amounts be devoted entirely to  In consumer theory this function can be used to illustrate how diminishing marginal usefulness can lead to a ceiling in purchases of certain goods.

**2.8** a. Because  is the derivative of   is an antiderivative of  By

the fundamental theorem of calculus,



where  is the fixed cost, which we will denote  for short. Rearranging,



b. By profit maximization, implying But  implies  Profit equals

 

If the firm is just breaking even, profit equals 0, implying fixed cost is



 c. When and follow the same steps as in part b, substituting

 fixed cost  Profit equals

 

 d. Assuming profit maximization, we have



e.

i. Using the above equation, 

ii. The envelope theorem states that  That is, the

derivative of the profit function yields this firm’s supply function.

Integrating over  shows the change in profits by the fundamental theorem of calculus:

 

**Analytical Problems**

**2.9 Concave and Quasiconcave Functions**

The proof is most easily accomplished through the use of the matrix algebra of quadratic forms. See, for example, Mas Colell *et al*., pp. 937–939. Intuitively, because concave functions lie below any tangent plane, their level curves must also be convex. But the converse is not true. Quasi-concave functions may exhibit “increasing returns to scale”; even though their level curves are convex, they may rise above the tangent plane when all variables are increased together.

 A counter example would be the Cobb-Douglas function which is always quasi-concave, but convex when 

**2.10** **The Cobb-Douglas Function**

 a. 

Clearly, all the terms in Equation 2.114 are negative.

b. A contour line is found by setting the function equal to a constant: implying Hence,



Further,



implying the countour line is convex.

c. Using Equation 2.98, which is negative for 

**2.11** **The Power Function**

 a. Since  and  the function is concave.

 b. Because  and  Equation 2.98 is satisfied; and the function is concave. Because  Equation 2.114 is also satisfied; so the function is quasi-concave.

 c. is quasi-concave as is  However,  is not concave for  This can be shown most easily by 

**2.12 Proof of Envelope Theorem**

a. The Lagrangian for this problem is

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 The first-order conditions are

 

 b.,c. Multiplication of each first order condition by the appropriate deriviative yields

 

 d. The optimal value of  is given by  Differentiation of this with respect to  shows how this optimal value changes with :

 

e. Differentiation of the constraint  yields

 

 f. Multiplying the results from part e by and using parts b and c yields

 

 This proves the envelope theorem.

g. In Example 2.8 we showed that  and that this hsows how much an extra unit of perimeter would raise the enclosed area. Direct differentiation of the Lagrangian in Equation 2.62 shows also that

 

This shows that the Lagrange multiplier does indeed show this incremental gain in this problem.

**2.13** **Taylor Approximations**

a. From Equation 2.85, a function in one variable is concave if  Using the quadratic Taylor formula to approximate this function at point :



The inequality holds because But the right hand side of this equation is the equation for the tangent to the function at point  So we have shown that any concave function must lie on or below the tangent to the function at that point.

 b. From Equation 2.98, a function in two variables is concave if 

 Hence the quadratic form  will also be negative. But

this says that the final portion of the Taylor expansion will be negative (by setting and ) and hence the function will be below its tangent plane.

**2.14** **More on Expected Value**

a. The tangent to at the point  will have the form  for all values of and  But, because the line  is above the function we know

 

 This proves Jensen’s inequality.

 b. Use the same procedure as in part a, but reverse the inequalities.

c. Let 

 

d. Use the hint to break up the integral defining expected value:



 e. 1. Show that this function integrates to 1:

 

 2. Calculate the cumulative distribution function:

 

 3. Using the result from part c:

 

4. To show Markov’s inequality use

 

f. 1. Show that the PDF integrates to 1:



 2. Calculate the expected value:



 3. Calculate ):



 4. All we must do is adjust the PDF so that it now sums to 1 over the

 new, smaller interval. Since 



 5. The expected value is again found through integration:

 

6. Eliminating the lowest values for *x* increases the expected value of the remaining values.

**2.15** **More on Variances**

 a. This is just an application of the definition of variance:

 

b. Here we let  and apply Markov’s inequality to  and remember that  can only take on positive values.



c. Let   be  independent random variables each with expected value  and variance 





Now, let 





d. Let  and 

 



Hence, variance in minimized for  In this case,  If  (not much of an increase).

e. Suppose that  and  Now







For example, if  then  and optimal diversification requires that the lower risk asset constitute two-thirds of the portfolio. Note, however, that it is still optimal to have some of the higher risk asset because asset returns are independent.

 **2.16** **More on Covariances**

 a. This is a direct result of the definition of covariance:

 

b. 

The final line is a result of Problems 2.15a and 2.16a.

c. The presence of the covariance term in the result of Problem 2.16b suggests that the results would differ. In the two variable case, however, this is not necessarily the situation. For example, suppose that *x* and *y* are identically distributed and that  $Cov(x,y)=rσ^{2}.$Using the prior notation,



The first order condition for a minimum is



implying



Regardless of the value of  With more than two random variables, however, covariances may indeed affect optimal weightings.

d. If  the correlation coefficient will be either  (if  is positive) or  (if  is negative) since  will factor out of the definition leaving only the ratio of the common variance of the two variables. With less than a perfect linear relationship 

e. If 



Hence,

