Chapter 1

Section 1.1

1.1.1 $\begin{bmatrix} x+2y=1 \\ 2x+3y=1 \end{bmatrix} \begin{matrix} \\ -2\times\text{1st equation} \end{matrix} \to \begin{bmatrix} x+2y & =1 \\ -y & =-1 \end{bmatrix} \begin{matrix} \\ \div(-1) \end{matrix} \to$

$\begin{bmatrix} x+2y=1 \\ y=1 \end{bmatrix} \begin{matrix} -2\times\text{2nd equation} \\ \\ \end{matrix} \to \begin{bmatrix} x & =-1 \\ y & =1 \end{bmatrix}$, so that $(x,y)=(-1,1)$.

1.1.2 $\begin{bmatrix} 4x+3y & =2 \\ 7x+5y & =3 \end{bmatrix} \begin{matrix} \div 4 \\ \\ \end{matrix} \to \begin{bmatrix} x+\frac{3}{4}y & =\frac{1}{2} \\ 7x+5y & =3 \end{bmatrix} \begin{matrix} \\ -7\times\text{1st equation} \end{matrix} \to$

$\begin{bmatrix} x+\frac{3}{4}y & =\frac{1}{2} \\ -\frac{1}{4}y & =-\frac{1}{2} \end{bmatrix} \begin{matrix} \\ \times(-4) \end{matrix} \to \begin{bmatrix} x+\frac{3}{4}y & =\frac{1}{2} \\ y & =2 \end{bmatrix} \begin{matrix} -\frac{3}{4}\times\text{2nd equation} \end{matrix} \to \begin{bmatrix} x & =-1 \\ y & =\ \ 2 \end{bmatrix}$,

so that $(x,y)=(-1,2)$.

1.1.3 $\begin{bmatrix} 2x+4y & =3 \\ 3x+6y & =2 \end{bmatrix} \begin{matrix} \div 2 \\ \\ \end{matrix} \to \begin{bmatrix} x+2y & =\frac{3}{2} \\ 3x+6y & =2 \end{bmatrix} \begin{matrix} \\ -3\times\text{1st equation} \end{matrix} \to \begin{bmatrix} x+2y & =\ \ \frac{3}{2} \\ 0 & =-\frac{5}{2} \end{bmatrix}$.

So there is no solution.

1.1.4 $\begin{bmatrix} 2x+4y & =2 \\ 3x+6y & =3 \end{bmatrix} \begin{matrix} \div 2 \\ \\ \end{matrix} \to \begin{bmatrix} x+2y & =1 \\ 3x+6y & =3 \end{bmatrix} \begin{matrix} \\ -3\times\text{1st equation} \end{matrix} \to \begin{bmatrix} x+2y & =1 \\ 0 & =0 \end{bmatrix}$

This system has infinitely many solutions: if we choose $y=t$, an arbitrary real number, then the equation $x+2y=1$ gives us $x=1-2y=1-2t$. Therefore the general solution is $(x,y)=(1-2t,t)$, where t is an arbitrary real number.

1.1.5 $\begin{bmatrix} 2x+3y & =0 \\ 4x+5y & =0 \end{bmatrix} \begin{matrix} \div 2 \\ \\ \end{matrix} \to \begin{bmatrix} x+\frac{3}{2}y & =0 \\ 4x+5y & =0 \end{bmatrix} \begin{matrix} \\ -4\times\text{1st equation} \end{matrix} \to$

$\begin{bmatrix} x+\frac{3}{2}y & =0 \\ -y & =0 \end{bmatrix} \begin{matrix} \\ \div(-1) \end{matrix} \to \begin{bmatrix} x+\frac{3}{2}y & =0 \\ y & =0 \end{bmatrix} \begin{matrix} -\frac{3}{2}\times\text{2nd equation} \\ \\ \end{matrix} \to \begin{bmatrix} x & =0 \\ y & =0 \end{bmatrix}$,

so that $(x,y)=(0,0)$.

1.1.6 $\begin{bmatrix} x+2y+3z & =\ \ 8 \\ x+3y+3z & =10 \\ x+2y+4z & =\ \ 9 \end{bmatrix} \begin{matrix} \\ -I \\ -I \end{matrix} \to \begin{bmatrix} x+2y+3z & =8 \\ y & =2 \\ z & =1 \end{bmatrix} \begin{matrix} -2(II) \\ \\ \\ \end{matrix} \to$

$\begin{bmatrix} x+3z & =4 \\ y & =2 \\ z & =1 \end{bmatrix} \begin{matrix} -3(III) \\ \\ \\ \end{matrix} \to \begin{bmatrix} x=1 \\ y=2 \\ z=1 \end{bmatrix}$, so that $(x,y,z)=(1,2,1)$.

1.1.7 $\begin{bmatrix} x+2y+3z & =1 \\ x+3y+4z & =3 \\ x+4y+5z & =4 \end{bmatrix} \begin{matrix} \\ -I \\ -I \end{matrix} \to \begin{bmatrix} x+2y+3z & =1 \\ y+z & =2 \\ 2y+2z & =3 \end{bmatrix} \begin{matrix} -2(II) \\ \\ -2(II) \end{matrix} \to \begin{bmatrix} x+z & =-3 \\ y+z & =\ \ 2 \\ 0 & =-1 \end{bmatrix}$

This system has no solution.

1.1.8 $$\begin{bmatrix} x+2y+3z & =0 \\ 4x+5y+6z & =0 \\ 7x+8y+10z & =0 \end{bmatrix} \begin{matrix} \\ -4(I) \\ -7(I) \end{matrix} \rightarrow \begin{bmatrix} x+2y+3z & =0 \\ -3y-6z & =0 \\ -6y-11z & =0 \end{bmatrix} \div(-3) \rightarrow$$

$$\begin{bmatrix} x+2y+3z & =0 \\ y+2z & =0 \\ -6y-11z & =0 \end{bmatrix} \begin{matrix} -2(II) \\ \\ +6(II) \end{matrix} \rightarrow \begin{bmatrix} x-z & =0 \\ y+2z & =0 \\ z & =0 \end{bmatrix} \begin{matrix} +III \\ -2(III) \\ \end{matrix} \rightarrow \begin{bmatrix} x & =0 \\ y & =0 \\ z & =0 \end{bmatrix},$$

so that $(x, y, z) = (0, 0, 0)$.

1.1.9 $$\begin{bmatrix} x+2y+3z & =1 \\ 3x+2y+z & =1 \\ 7x+2y-3z & =1 \end{bmatrix} \begin{matrix} \\ -3(I) \\ -7(I) \end{matrix} \rightarrow \begin{bmatrix} x+2y+3z & =1 \\ -4y-8z & =-2 \\ -12y-24z & =-6 \end{bmatrix} \div(-4) \rightarrow$$

$$\begin{bmatrix} x+2y+3z & =1 \\ y+2z & =\frac{1}{2} \\ -12y-24z & =-6 \end{bmatrix} \begin{matrix} -2(II) \\ \\ +12(II) \end{matrix} \rightarrow \begin{bmatrix} x-z & =0 \\ y+2z & =\frac{1}{2} \\ 0 & =0 \end{bmatrix}$$

This system has infinitely many solutions: if we choose $z = t$, an arbitrary real number, then we get $x = z = t$ and $y = \frac{1}{2} - 2z = \frac{1}{2} - 2t$. Therefore, the general solution is $(x, y, z) = \left(t, \frac{1}{2} - 2t, t\right)$, where t is an arbitrary real number.

1.1.10 $$\begin{bmatrix} x+2y+3z & =1 \\ 2x+4y+7z & =2 \\ 3x+7y+11z & =8 \end{bmatrix} \begin{matrix} \\ -2(I) \\ -3(I) \end{matrix} \rightarrow \begin{bmatrix} x+2y+3z & =1 \\ z & =0 \\ y+2z & =5 \end{bmatrix} \begin{matrix} Swap: \\ II \leftrightarrow III \end{matrix} \rightarrow$$

$$\begin{bmatrix} x+2y+3z & =1 \\ y+2z & =5 \\ z & =0 \end{bmatrix} \begin{matrix} -2(II) \\ \\ \end{matrix} \rightarrow \begin{bmatrix} x-z & =-9 \\ y+2z & =5 \\ z & =0 \end{bmatrix} \begin{matrix} +III \\ -2(III) \\ \end{matrix} \rightarrow \begin{bmatrix} x & =-9 \\ y & =5 \\ z & =0 \end{bmatrix},$$

so that $(x, y, z) = (-9, 5, 0)$.

1.1.11 $$\begin{bmatrix} x-2y & =2 \\ 3x+5y & =17 \end{bmatrix} \begin{matrix} \\ -3(I) \end{matrix} \rightarrow \begin{bmatrix} x-2y & =2 \\ 11y & =11 \end{bmatrix} \div 11 \rightarrow \begin{bmatrix} x-2y & =2 \\ y & =1 \end{bmatrix} \begin{matrix} +2(II) \\ \end{matrix} \rightarrow \begin{bmatrix} x & =4 \\ y & =1 \end{bmatrix},$$

so that $(x, y) = (4, 1)$. See Figure 1.1.

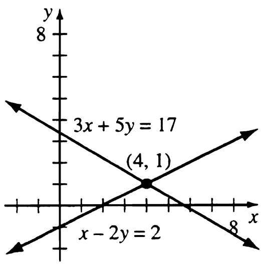


Figure 1.1: for Problem 1.1.11.

1.1.12 $$\begin{bmatrix} x-2y & =3 \\ 2x-4y & =6 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix} \rightarrow \begin{bmatrix} x-2y & =3 \\ 0 & =0 \end{bmatrix}$$

This system has infinitely many solutions: If we choose $y = t$, an arbitrary real number, then the equation $x - 2y = 3$ gives us $x = 3 + 2y = 3 + 2t$. Therefore the general solution is $(x, y) = (3 + 2t, t)$, where t is an arbitrary real number. (See Figure 1.2.)

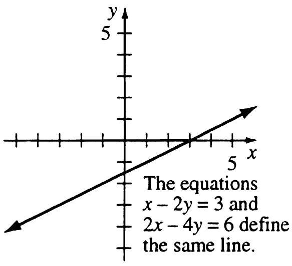


Figure 1.2: for Problem 1.1.12.

1.1.**13** $\begin{bmatrix} x - 2y & = 3 \\ 2x - 4y & = 8 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix} \rightarrow \begin{bmatrix} x - 2y & = 3 \\ 0 & = 2 \end{bmatrix}$, which has no solutions. (See Figure 1.3.)

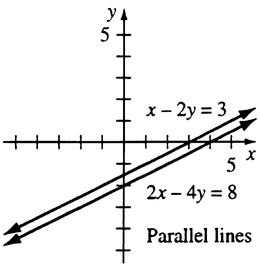


Figure 1.3: for Problem 1.1.13.

1.1.**14** The system reduces to $\begin{bmatrix} x + 5z & = 0 \\ y - z & = 0 \\ 0 & = 1 \end{bmatrix}$, so that there is no solution; no point in space belongs to all three planes.

Compare with Figure 2b.

1.1.**15** The system reduces to $\begin{bmatrix} x & = 0 \\ y & = 0 \\ z & = 0 \end{bmatrix}$ so the unique solution is $(x, y, z) = (0, 0, 0)$. The three planes intersect at the origin.

1.1.**16** The system reduces to $\begin{bmatrix} x + 5z & = 0 \\ y - z & = 0 \\ 0 & = 0 \end{bmatrix}$, so the solutions are of the form $(x, y, z) = (-5t, t, t)$, where t is an arbitrary number. The three planes intersect in a line; compare with Figure 2a.

1.1.**17** $\begin{bmatrix} x + 2y & = a \\ 3x + 5y & = b \end{bmatrix} \begin{matrix} \\ -3(I) \end{matrix} \rightarrow \begin{bmatrix} x + 2y & = a \\ -y & = -3a + b \end{bmatrix} \begin{matrix} \\ \div(-1) \end{matrix} \rightarrow \begin{bmatrix} x + 2y & = a \\ y & = 3a - b \end{bmatrix} \begin{matrix} -2(II) \\ \\ \end{matrix}$

$\begin{bmatrix} x & = -5a+2b \\ y & = 3a-b \end{bmatrix}$, so that $(x,y) = (-5a+2b,\ 3a-b)$.

1.1.**18** $\begin{bmatrix} x+2y+3z & = a \\ x+3y+8z & = b \\ x+2y+2z & = c \end{bmatrix} \begin{matrix} \\ -I \\ -I \end{matrix} \rightarrow \begin{bmatrix} x+2y+3z & = a \\ y+5z & = -a+b \\ -z & = -a+c \end{bmatrix} \begin{matrix} -2(II) \\ \\ \\ \end{matrix} \rightarrow$

$\begin{bmatrix} x-7z & = 3a-2b \\ y+5z & = -a+b \\ -z & = -a+c \end{bmatrix} \begin{matrix} \\ \\ \div(-1) \end{matrix} \rightarrow \begin{bmatrix} x-7z & = 3a-2b \\ y+5z & = -a+b \\ z & = a-c \end{bmatrix} \begin{matrix} +7(III) \\ -5(III) \\ \\ \end{matrix} \rightarrow \begin{bmatrix} x & = 10a-2b-7c \\ y & = -6a+b+5c \\ z & = a-c \end{bmatrix}$,

so that $(x,y,z) = (10a-2b-7c,\ -6a+b+5c,\ a-c)$.

1.1.**19** The system reduces to $\begin{bmatrix} x+z & = 1 \\ y-2z & = -3 \\ 0 & = k-7 \end{bmatrix}$.

a. The system has solutions if $k-7=0$, or $k=7$.

b. If $k=7$ then the system has infinitely many solutions.

c. If $k=7$ then we can choose $z=t$ freely and obtain the solutions

$(x,y,z) = (1-t, -3+2t, t)$.

1.1.**20** The system reduces to $\begin{bmatrix} x-3z & = & 1 \\ y+2z & = & 1 \\ (k^2-4)z & = & k-2 \end{bmatrix}$

This system has a unique solution if $k^2-4 \neq 0$, that is, if $k \neq \pm 2$.

If $k=2$, then the last equation is $0=0$, and there will be infinitely many solutions.

If $k=-2$, then the last equation is $0=-4$, and there will be no solutions.

1.1.**21** Let x, y, and z represent the three numbers we seek. We set up a system of equations and solve systematically (although there are short cuts):

$$\left| \begin{matrix} x & +y & & = 24 \\ x & & +z & = 28 \\ & y+ & z & = 30 \end{matrix} \right| \begin{matrix} \\ -(I) \\ \\ \end{matrix} \rightarrow \left| \begin{matrix} x & +y & & = 24 \\ & -y & +z & = 4 \\ & y+ & z & = 30 \end{matrix} \right| \begin{matrix} \\ \div(-1) \\ \\ \end{matrix} \rightarrow \left| \begin{matrix} x & +y & & = 24 \\ & y & -z & = -4 \\ & y+ & z & = 30 \end{matrix} \right| \begin{matrix} -(II) \\ \\ -(II) \end{matrix}$$

$$\rightarrow \left| \begin{matrix} x & & +z & = 28 \\ & y & -z & = -4 \\ & & 2z & = 34 \end{matrix} \right| \begin{matrix} \\ \\ \div 2 \end{matrix} \rightarrow \left| \begin{matrix} x & & +z & = 28 \\ & y & -z & = -4 \\ & & z & = 17 \end{matrix} \right| \begin{matrix} -(III) \\ +(III) \\ \\ \end{matrix} \rightarrow \left| \begin{matrix} x & & & = 11 \\ & y & & = 13 \\ & & z & = 17 \end{matrix} \right|$$

We see that $x=11,\ y=13,$ and $z=17$.

1.1.**22** Let $x =$ the number of male children and $y =$ the number of female children.

Then the statement "Emile has twice as many sisters as brothers" translates into

$y = 2(x-1)$ and "Gertrude has as many brothers as sisters" translates into

$x = y - 1$.

Solving the system $\begin{bmatrix} -2x + y & = -2 \\ x - y & = -1 \end{bmatrix}$ gives $x = 3$ and $y = 4$.

There are seven children in this family.

1.1.**23** a Note that the demand D_1 for product 1 increases with the increase of price P_2; likewise the demand D_2 for product 2 increases with the increase of price P_1. This indicates that the two products are competing; some people will switch if one of the products gets more expensive.

b Setting $D_1 = S_1$ and $D_2 = S_2$ we obtain the system $\begin{bmatrix} 70 - 2P_1 + P_2 & = -14 + 3P_1 \\ 105 + P_1 - P_2 & = -7 + 2P_2 \end{bmatrix}$, or $\begin{bmatrix} -5P_1 + P_2 & = -84 \\ P_1 - 3P_2 & = 112 \end{bmatrix}$, which yields the unique solution $P_1 = 26$ and $P_2 = 46$.

1.1.**24** The total demand for the product of Industry A is 1000 (the consumer demand) plus 0.1b (the demand from Industry B). The output a must meet this demand: $a = 1000 + 0.1b$.

Setting up a similar equation for Industry B we obtain the system $\begin{bmatrix} a & = 1000 + 0.1b \\ b & = 780 + 0.2a \end{bmatrix}$ or $\begin{bmatrix} a - 0.1b & = 1000 \\ -0.2a + b & = 780 \end{bmatrix}$, which yields the unique solution $a = 1100$ and $b = 1000$.

1.1.**25** The total demand for the products of Industry A is 310 (the consumer demand) plus $0.3b$ (the demand from Industry B). The output a must meet this demand: $a = 310 + 0.3b$.

Setting up a similar equation for Industry B we obtain the system $\begin{bmatrix} a & = 310 + 0.3b \\ b & = 100 + 0.5a \end{bmatrix}$ or $\begin{bmatrix} a - 0.3b & = 310 \\ -0.5a + b & = 100 \end{bmatrix}$, which yields the solution $a = 400$ and $b = 300$.

1.1.**26** Since $x(t) = a\sin(t) + b\cos(t)$ we can compute $\frac{dx}{dt} = a\cos(t) - b\sin(t)$ and $\frac{d^2x}{dt^2}$

$= -a\sin(t) - b\cos(t)$. Substituting these expressions into the equation $\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t)$ and simplifying gives $(b - 2a)\sin(t) + (-a - 2b)\cos(t) = \cos(t)$. Comparing the coefficients of $\sin(t)$ and $\cos(t)$ on both sides of the equation then yields the system $\begin{bmatrix} -2a + b & = 0 \\ -a - 2b & = 1 \end{bmatrix}$, so that $a = -\frac{1}{5}$ and $b = -\frac{2}{5}$. See Figure 1.4.

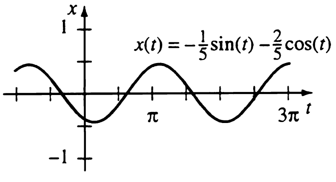


Figure 1.4: for Problem 1.1.26.

1.1.**27** a Substituting $\lambda = 5$ yields the system

$$\begin{bmatrix} 7x - y & = 5x \\ -6x + 8y & = 5y \end{bmatrix} \text{ or } \begin{bmatrix} 2x - y & = 0 \\ -6x + 3y & = 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2x - y & = 0 \\ 0 & = 0 \end{bmatrix}.$$

There are infinitely many solutions, of the form $(x, y) = \left(\frac{t}{2}, t\right)$, where t is an arbitrary real number.

b Proceeding as in part (a), we find $(x, y) = \left(-\frac{1}{3}t, t\right)$.

c Proceedings as in part (a), we find only the solution $(0, 0)$.

1.1.**28** Use the distance from Stein to Schaffhausen (and from Stein to Constance) as the unit of length.

Let the speed of the boat and the speed of the river flow be v_b and v_s, respectively.

Using the formula speed $= \frac{\text{distance}}{\text{time}}$ and measuring time in hours we get:

$v_b - v_s = 1$

$v_b + v_s = \frac{3}{2}$

We find $v_b = \frac{5}{4}$ and $v_s = \frac{1}{4}$. The time it takes to travel from Stein to Constance is time $= \frac{\text{distance}}{\text{speed}} = \frac{1}{v_b} = \frac{4}{5}$ hours, or 48 minutes.

1.1.**29** Let v be the speed of the boat relative to the water, and s be the speed of the stream; then the speed of the boat relative to the land is $v + s$ downstream and $v - s$ upstream. Using the fact that (distance) = (speed)(time), we obtain the system

$$\begin{bmatrix} 8 & = (v + s)\frac{1}{3} \\ 8 & = (v - s)\frac{2}{3} \end{bmatrix} \quad \begin{matrix} \leftarrow \text{ downstream} \\ \leftarrow \text{ upstream} \end{matrix}$$

The solution is $v = 18$ and $s = 6$.

1.1.**30** The thermal equilibrium condition requires that $T_1 = \frac{T_2+200+0+0}{4}$, $T_2 = \frac{T_1+T_3+200+0}{4}$, and $T_3 = \frac{T_2+400+0+0}{4}$.

We can rewrite this system as $\begin{bmatrix} -4T_1 + T_2 & = -200 \\ T_1 - 4T_2 + T_3 & = -200 \\ T_2 - 4T_3 & = -400 \end{bmatrix}$

The solution is $(T_1, T_2, T_3) = (75, 100, 125)$.

1.1.**31** To assure that the graph goes through the point $(1, -1)$, we substitute $t = 1$ and $f(t) = -1$ into the equation $f(t) = a + bt + ct^2$ to give $-1 = a + b + c$.

Proceeding likewise for the two other points, we obtain the system $\begin{bmatrix} a + b + c & = -1 \\ a + 2b + 4c & = 3 \\ a + 3b + 9c & = 13 \end{bmatrix}$.

The solution is $a = 1,\ b = -5$, and $c = 3$, and the polynomial is $f(t) = 1 - 5t + 3t^2$. (See Figure 1.5.)

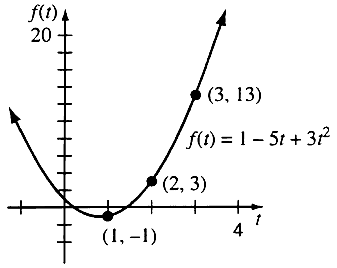


Figure 1.5: for Problem 1.1.31.

1.1.**32** Proceeding as in the previous exercise, we obtain the system $\left[\begin{array}{ll|} a+b+c & =p \\ a+2b+4c & =q \\ a+3b+9c & =r \end{array}\right]$.

The unique solution is $\begin{bmatrix} a & =3p-3q+r \\ b & =-2.5p+4q-1.5r \\ c & =0.5p-q+0.5r \end{bmatrix}$.

Only one polynomial of degree 2 goes through the three given points, namely,

$f(t) = 3p - 3q + r + (-2.5p + 4q - 1.5r)t + (0.5p - q + 0.5r)t^2$.

1.1.**33** $f(t)$ is of the form $at^2 + bt + c$. So $f(1) = a(1^2) + b(1) + c = 3$, and $f(2) = a(2^2) + b(2) + c = 6$. Also, $f'(t) = 2at + b$, meaning that $f'(1) = 2a + b = 1$.

So we have a system of equations: $\begin{bmatrix} a+b+c=3 \\ 4a+2b+c=6 \\ 2a+b=1 \end{bmatrix}$

which reduces to $\begin{bmatrix} a=2 \\ b=-3 \\ c=4 \end{bmatrix}$.

Thus, $f(t) = 2t^2 - 3t + 4$ is the only solution.

1.1.**34** $f(t)$ is of the form $at^2 + bt + c$. So, $f(1) = a(1^2) + b(1) + c = 1$ and $f(2) = 4a + 2b + c = 0$. Also, $\int_1^2 f(t)dt = \int_1^2 (at^2 + bt + c)dt$

$= \frac{a}{3}t^3 + \frac{b}{2}t^2 + ct|_1^2$

$= \frac{8}{3}a + 2b + 2c - (\frac{a}{3} + \frac{b}{2} + c)$

$= \frac{7}{3}a + \frac{3}{2}b + c = -1$.

So we have a system of equations: $\begin{bmatrix} a+b+c=1 \\ 4a+2b+c=0 \\ \frac{7}{3}a+\frac{3}{2}b+c=-1 \end{bmatrix}$

which reduces to $\begin{bmatrix} a=9 \\ b=-28 \\ c=20 \end{bmatrix}$.

Thus, $f(t) = 9t^2 - 28t + 20$ is the only solution.

1.1.**35** $f(t)$ is of the form $at^2 + bt + c$. $f(1) = a + b + c = 1$, $f(3) = 9a + 3b + c = 3$, and $f'(t) = 2at + b$, so $f'(2) = 4a + b = 1$.

Now we set up our system to be $\begin{bmatrix} a+b+c=1 \\ 9a+3b+c=3 \\ 4a+b=1 \end{bmatrix}$.

This reduces to $\begin{bmatrix} a - \frac{c}{3} = 0 \\ b + \frac{4}{3}c = 1 \\ 0 = 0 \end{bmatrix}$.

We write everything in terms of a, revealing $c = 3a$ and $b = 1 - 4a$.

So, $f(t) = at^2 + (1 - 4a)t + 3a$ for an arbitrary a.

1.1.**36** $f(t) = at^2 + bt + c$, so $f(1) = a + b + c = 1$, $f(3) = 9a + 3b + c = 3$. Also, $f'(2) = 3$, so $2(2)a + b = 4a + b = 3$.

Thus, our system is $\begin{bmatrix} a+b+c=1 \\ 9a+3b+c=3 \\ 4a+b=3 \end{bmatrix}$.

When we reduce this, however, our last equation becomes $0 = 2$, meaning that this system is inconsistent.

1.1.**37** $f(t) = ae^{3t} + be^{2t}$, so $f(0) = a + b = 1$ and $f'(t) = 3ae^{3t} + 2be^{2t}$, so $f'(0) = 3a + 2b = 4$.

Thus we obtain the system $\begin{bmatrix} a+b=1 \\ 3a+2b=4 \end{bmatrix}$,

which reveals $\begin{bmatrix} a=2 \\ b=-1 \end{bmatrix}$.

So $f(t) = 2e^{3t} - e^{2t}$.

1.1.**38** $f(t) = a\cos(2t) + b\sin(2t)$ and $3f(t) + 2f'(t) + f''(t) = 17\cos(2t)$.

$f'(t) = 2b\cos(2t) - 2a\sin(2t)$ and $f''(t) = -4b\sin(2t) - 4a\cos(2t)$.

So, $17\cos(2t) = 3(a\cos(2t) + b\sin(2t)) + 2(2b\cos(2t) - 2a\sin(2t)) + (-4b\sin(2t) - 4a\cos(2t)) = (-4a + 4b + 3a)\cos(2t) + (-4b - 4a + 3b)\sin(2t) = (-a + 4b)\cos(2t) + (-4a - b)\sin(2t)$.

So, our system is: $\begin{bmatrix} -a+4b=17 \\ -4a-b=0 \end{bmatrix}$.

This reduces to: $\begin{bmatrix} a=-1 \\ b=4 \end{bmatrix}$.

So our function is $f(t) = -\cos(2t) + 4\sin(2t)$.

1.1.**39** Plugging the three points (x, y) into the equation $a + bx + cy + x^2 + y^2 = 0$, leads to a system of linear equations for the three unknowns (a, b, c).

$$\begin{aligned} a + 5b + 5c + 25 + 25 &= 0 \\ a + 4b + 6c + 16 + 36 &= 0 \\ a + 6b + 2c + 36 + 4 &= 0\,. \end{aligned}$$

The solution is $a = -20, b = -2, c = -4$. $-20 - 2x - 4y + x^2 + y^2 = 0$ is a circle of radius 5 centered at $(1, 2)$.

1.1.**40** Plug the three points into the equation $ax^2 + bxy + cy^2 = 1$. We obtain a system of linear equations

$$\begin{aligned} a + 2b + 4c &= 1 \\ 4a + 4b + 4c &= 1 \\ 9a + 3b + c &= 1\,. \end{aligned}$$

The solution is $a = 3/20, b = -9/40, c = 13/40$. This is the ellipse $(3/20)x^2 - (9/40)xy + (13/40)y^2 = 1$.

1.1.**41** The given system reduces to $\begin{bmatrix} x - z & = \frac{-5a+2b}{3} \\ y + 2z & = \frac{4a-b}{3} \\ 0 & = a - 2b + c \end{bmatrix}$.

This system has solutions (in fact infinitely many) if $a - 2b + c = 0$.

The points (a, b, c) with this property form a plane through the origin.

1.1.**42** a $x_1 = -3$

$x_2 = 14 + 3x_1 = 14 + 3(-3) = 5$

$x_3 = 9 - x_1 - 2x_2 = 9 + 3 - 10 = 2$

$x_4 = 33 + x_1 - 8x_2 + 5x_3 - x_4 = 33 - 3 - 40 + 10 = 0$,

so that $(x_1, x_2, x_3, x_4) = (-3, 5, 2, 0)$.

b $x_4 = 0$

$x_3 = 2 - 2x_4 = 2$

$x_2 = 5 - 3x_3 - 7x_4 = 5 - 6 = -1$

$x_1 = -3 - 2x_2 + x_3 - 4x_4 = -3 + 2 + 2 = 1$,

so that $(x_1, x_2, x_3, x_4) = (1, -1, 2, 0)$.

1.1.**43** a The two lines intersect unless $t = 2$ (in which case both lines have slope -1).
To draw a rough sketch of $x(t)$, note that
$\lim_{t\to\infty} x(t) = \lim_{t\to-\infty} x(t) = -1$ (the line $x + \frac{t}{2}y = t$ becomes almost horizontal)
and
$\lim_{t\to 2^-} x(t) = \infty$, $\lim_{t\to 2^+} x(t) = -\infty$.

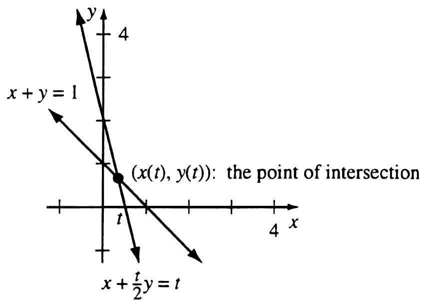


Figure 1.6: for Problem 1.1.43a.

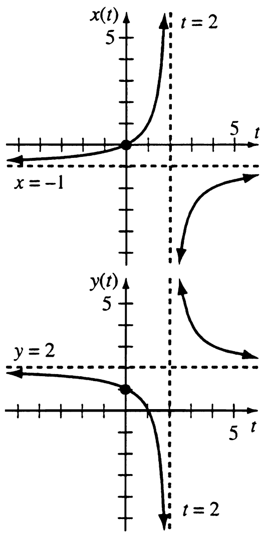


Figure 1.7: for Problem 1.1.43a.

Also note that $x(t)$ is positive if t is between 0 and 2, and negative otherwise.
Apply similar reasoning to $y(t)$. (See Figures 1.6 and 1.7.)

b $x(t) = \frac{-t}{t-2}$, and $y(t) = \frac{2t-2}{t-2}$.

1.1.**44** We can think of the line through the points $(1, 1, 1)$ and $(3, 5, 0)$ as the intersection of any two planes through these two points; each of these planes will be defined by an equation of the form $ax + by + cz = d$. It is required that $1a + 1b + 1c = d$ and $3a + 5b + 0c = d$.

Now the system $\begin{bmatrix} a & +b & +c & -d & = & 0 \\ 3a & +5b & & -d & = & 0 \end{bmatrix}$ reduces to

$$\begin{bmatrix} a & & +\frac{5}{2}c & -2d & = & 0 \\ & b & -\frac{3}{2}c & +d & = & 0 \end{bmatrix}.$$

We can choose arbitrary real numbers for c and d; then $a = -\frac{5}{2}c + 2d$ and $b = \frac{3}{2}c - d$. For example, if we choose $c = 2$ and $d = 0$, then $a = -5$ and $b = 3$, leading to the equation $-5x + 3y + 2z = 0$. If we choose $c = 0$ and $d = 1$, then $a = 2$ and $b = -1$, giving the equation $2x - y = 1$.

We have found one possible answer: $\begin{bmatrix} -5x & +3y & +2z & = & 0 \\ 2x & -y & & = & 1 \end{bmatrix}$.

1.1.**45** To eliminate the arbitrary constant t, we can solve the last equation for t to give $t = z - 2$, and substitute $z - 2$ for t in the first two equations, obtaining $\begin{bmatrix} x & = 6 + 5(z-2) \\ y & = 4 + 3(z-2) \end{bmatrix}$ or $\begin{bmatrix} x - 5z & = -4 \\ y - 3z & = -2 \end{bmatrix}$.

This system does the job.

1.1.**46** Let $b =$ Boris' money, $m =$ Marina's money, and $c =$ cost of a chocolate bar.

We are told that $\begin{bmatrix} b + \frac{1}{2}m & = c \\ \frac{1}{2}b + m & = 2c \end{bmatrix}$, with solution $(b, m) = (0, 2c)$.

Boris has no money.

1.1.**47** Let us start by reducing the system:

$$\begin{bmatrix} x + 2y + 3z & = 39 \\ x + 3y + 2z & = 34 \\ 3x + 2y + z & = 26 \end{bmatrix} \begin{matrix} \\ -I \\ -3(I) \end{matrix} \rightarrow \begin{bmatrix} x + 2y + 3z & = 39 \\ y - z & = -5 \\ -4y - 8z & = -91 \end{bmatrix}$$

Note that the last two equations are exactly those we get when we substitute

$x = 39 - 2y - 3z$: either way, we end up with the system $\begin{bmatrix} y - z & = -5 \\ -4y - 8z & = -91 \end{bmatrix}$.

1.1.**48** a We set up two equations here, with our variables: $x_1 =$ servings of rice, $x_2 =$ servings of yogurt.

So our system is: $\begin{bmatrix} 3x_1 & +12x_2 & = 60 \\ 30x_1 & +20x_2 & = 300 \end{bmatrix}$.

Solving this system reveals that $x_1 = 8, x_2 = 3$.

b Again, we set up our equations: $\begin{bmatrix} 3x_1 & +12x_2 & = P \\ 30x_1 & +20x_2 & = C \end{bmatrix}$,

and reduce them to find that $x_1 = -\frac{P}{15} + \frac{C}{25}$, while $x_2 = \frac{P}{10} - \frac{C}{100}$.

1.1.**49** Let $x_1 =$ number of one-dollar bills, $x_2 =$ the number of five-dollar bills, and $x_3 =$ the number of ten-dollar bills. Then our system looks like: $\begin{bmatrix} x_1 + x_2 + x_3 & = 32 \\ x_1 + 5x_2 + 10x_3 & = 100 \end{bmatrix}$,

which reduces to give us solutions that fit: $x_1 = 15 + \frac{5}{4}x_3$, $x_2 = 17 - \frac{9}{4}x_3$, where x_3 can be chosen freely. Now let's keep in mind that x_1, x_2, and x_3 must be positive integers and see what conditions this imposes on the variable x_3. We see that since x_1 and x_2 must be integers, x_3 must be a multiple of 4. Furthermore, x_3 must be positive, and $x_2 = 17 - \frac{9}{4}x_3$ must be positive as well, meaning that $x_3 < \frac{68}{9}$. These constraints leave us with only one possibility, $x_3 = 4$, and we can compute the corresponding values $x_1 = 15 + \frac{5}{4}x_3 = 20$ and $x_2 = 17 - \frac{9}{4}x_3 = 8$.

Thus, we have 20 one-dollar bills, 8 five-dollar bills, and 4 ten-dollar bills.

1.1.**50** Let x_1, x_2, x_3 be the number of 20 cent, 50 cent, and 2 Euro coins, respectively. Then we need solutions to the system: $\begin{bmatrix} x_1 & +x_2 & +x_3 & = 1000 \\ .2x_1 & +.5x_2 & +2x_3 & = 1000 \end{bmatrix}$

this system reduces to: $\begin{bmatrix} x_1 & & -5x_3 & = -\frac{5000}{3} \\ & x_2 & +6x_3 & = \frac{8000}{3} \end{bmatrix}$.

Our solutions are then of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 - \frac{5000}{3} \\ -6x_3 + \frac{8000}{3} \\ x_3 \end{bmatrix}$. Unfortunately for the meter maids, there are no integer solutions to this problem. If x_3 is an integer, then neither x_1 nor x_2 will be an integer, and no one will ever claim the Ferrari.

Section 1.2

1.2.**1** $\begin{bmatrix} 1 & 1 & -2 & \vdots & 5 \\ 2 & 3 & 4 & \vdots & 2 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix} \to \begin{bmatrix} 1 & 1 & -2 & \vdots & 5 \\ 0 & 1 & 8 & \vdots & -8 \end{bmatrix} \begin{matrix} -II \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 0 & -10 & \vdots & 13 \\ 0 & 1 & 8 & \vdots & -8 \end{bmatrix}$

$\begin{bmatrix} x - 10z & = 13 \\ y + 8z & = -8 \end{bmatrix} \longrightarrow \begin{bmatrix} x & = 13 + 10z \\ y & = -8 - 8z \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 + 10t \\ -8 - 8t \\ t \end{bmatrix}$, where t is an arbitrary real number.

1.2.**2** $\begin{bmatrix} 3 & 4 & -1 & \vdots & 8 \\ 6 & 8 & -2 & \vdots & 3 \end{bmatrix} \begin{matrix} \div 3 \\ \\ \end{matrix} \to \begin{bmatrix} 1 & \frac{4}{3} & -\frac{1}{3} & \vdots & \frac{8}{3} \\ 6 & 8 & -2 & \vdots & 3 \end{bmatrix} \begin{matrix} \\ -6(I) \end{matrix} \to \begin{bmatrix} 1 & \frac{4}{3} & -\frac{1}{3} & \vdots & \frac{8}{3} \\ 0 & 0 & 0 & \vdots & -13 \end{bmatrix}$

This system has no solutions, since the last row represents the equation $0 = -13$.

1.2.**3** $x = 4 - 2y - 3z$

y and z are free variables; let $y = s$ and $z = t$.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - 2s - 3t \\ s \\ t \end{bmatrix}$, where s and t are arbitrary real numbers.

1.2.**4** $\begin{bmatrix} 1 & 1 & \vdots & 1 \\ 2 & -1 & \vdots & 5 \\ 3 & 4 & \vdots & 2 \end{bmatrix} \begin{matrix} \\ -2(I) \\ -3(I) \end{matrix} \to \begin{bmatrix} 1 & 1 & \vdots & 1 \\ 0 & -3 & \vdots & 3 \\ 0 & 1 & \vdots & -1 \end{bmatrix} \begin{matrix} \\ \div(-3) \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 1 & \vdots & 1 \\ 0 & 1 & \vdots & -1 \\ 0 & 1 & \vdots & -1 \end{bmatrix} \begin{matrix} -II \\ \\ -II \end{matrix} \to$

$\begin{bmatrix} 1 & 0 & \vdots & 2 \\ 0 & 1 & \vdots & -1 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$, so that $\begin{matrix} x & = & 2 \\ y & = & -1 \end{matrix}$.

1.2.**5** $\begin{bmatrix} 0 & 0 & 1 & 1 & \vdots & 0 \\ 0 & 1 & 1 & 0 & \vdots & 0 \\ 1 & 1 & 0 & 0 & \vdots & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} swap: \\ I \leftrightarrow III \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ -I \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 \\ 0 & -1 & 0 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} -II \\ \\ \\ +II \end{matrix} \rightarrow$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & \vdots & 0 \\ 0 & 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} +III \\ -III \\ \\ -III \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 0 & -1 & \vdots & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & & & + & x_4 & = & 0 \\ & x_2 & & - & x_4 & = & 0 \\ & & x_3 & + & x_4 & = & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 & = -x_4 \\ x_2 & = x_4 \\ x_3 & = -x_4 \end{bmatrix}$$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix}$, where t is an arbitrary real number.

1.2.**6** The system is in rref already.

$$\begin{bmatrix} x_1 & = 3 + 7x_2 - x_5 \\ x_3 & = 2 + 2x_5 \\ x_4 & = 1 - x_5 \end{bmatrix}$$

Let $x_2 = t$ and $x_5 = r$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 + 7t - r \\ t \\ 2 + 2r \\ 1 - r \\ r \end{bmatrix}$$

1.2.**7** $\begin{bmatrix} 1 & 2 & 0 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 1 & 3 & 2 & \vdots & 0 \\ 0 & 0 & 1 & 4 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} \\ \\ -II \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 1 & 3 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 1 & -3 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} -2(III) \\ -3(III) \\ \\ \\ \end{matrix} \rightarrow$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 9 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 11 & \vdots & 0 \\ 0 & 0 & 0 & 1 & -3 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} -9(IV) \\ -11(IV) \\ +3(IV) \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 & = 0 \\ x_3 & = 0 \\ x_4 & = 0 \\ x_5 & = 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 & = -2x_2 \\ x_3 & = 0 \\ x_4 & = 0 \\ x_5 & = 0 \end{bmatrix}$$

Let $x_2 = t$. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix}$, where t is an arbitrary real number.

1.2.**8** $\begin{bmatrix} 0 & 1 & 0 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 0 & 4 & 8 & \vdots & 0 \end{bmatrix} \begin{matrix} \\ \div 4 \end{matrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & \vdots & 0 \end{bmatrix} \begin{matrix} -2(II) \\ \\ \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & \vdots & 0 \end{bmatrix}$

$\begin{bmatrix} x_2 - x_5 & = 0 \\ x_4 + 2x_5 & = 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_2 & = x_5 \\ x_4 & = -2x_5 \end{bmatrix}$

Let $x_1 = r,\ x_3 = s,\ x_5 = t$.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r \\ t \\ s \\ -2t \\ t \end{bmatrix}$, where r, t and s are arbitrary real numbers.

1.2.**9** $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 & \vdots & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & \vdots & 2 \end{bmatrix} \begin{matrix} swap: \\ I \leftrightarrow II \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & \vdots & 2 \\ 1 & 2 & 2 & 0 & -2 & 1 & \vdots & 2 \end{bmatrix} \begin{matrix} \\ \\ -I \end{matrix} \rightarrow$

$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & \vdots & 2 \\ 0 & 0 & 2 & 0 & -2 & 2 & \vdots & 2 \end{bmatrix} \begin{matrix} swap: \\ II \leftrightarrow III \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 2 & 0 & -2 & 2 & \vdots & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & \vdots & 2 \end{bmatrix} \div 2 \rightarrow$

$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & \vdots & 2 \end{bmatrix}$

$\begin{bmatrix} x_1 + 2x_2 + x_5 - x_6 & = 0 \\ x_3 - x_5 + x_6 & = 1 \\ x_4 + 2x_5 - x_6 & = 2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 & = -2x_2 - x_5 + x_6 \\ x_3 & = 1 + x_5 - x_6 \\ x_4 & = 2 - 2x_5 + x_6 \end{bmatrix}$

Let $x_2 = r,\ x_5 = s$, and $x_6 = t$.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2r - s + t \\ r \\ 1 + s - t \\ 2 - 2s + t \\ s \\ t \end{bmatrix}$, where r, s and t are arbitrary real numbers.

1.2.**10** The system reduces to $\begin{bmatrix} x_1 & & & + & x_4 & = & 1 \\ & x_2 & & - & 3x_4 & = & 2 \\ & & x_3 & + & 2x_4 & = & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 & = 1 - x_4 \\ x_2 & = 2 + 3x_4 \\ x_3 & = -3 - 2x_4 \end{bmatrix}$

Let $x_4 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t \\ 2+3t \\ -3-2t \\ t \end{bmatrix}$$, where t is an arbitrary real number.

1.2.**11** The system reduces to $\begin{bmatrix} x_1 & & + & 2x_3 & & = & 0 \\ & x_2 & - & 3x_3 & & = & 4 \\ & & & & x_4 & = & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 & = -2x_3 \\ x_2 & = 4+3x_3 \\ x_4 & = -2 \end{bmatrix}$.

Let $x_3 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 4+3t \\ t \\ -2 \end{bmatrix}$$

1.2.**12** The system reduces to $\begin{bmatrix} x_1 & & & & + & 3.5x_5 & + & x_6 & = & 0 \\ & x_2 & & & + & x_5 & & & = & 0 \\ & & x_3 & & & & - & \frac{5}{3}x_6 & = & 0 \\ & & & x_4 & + & 3x_5 & + & x_6 & = & 0 \end{bmatrix} \longrightarrow$

$$\begin{bmatrix} x_1 & = -3.5x_5 - x_6 \\ x_2 & = -x_5 \\ x_3 & = \frac{5}{3}x_6 \\ x_4 & = -3x_5 - x_6 \end{bmatrix}.$$

Let $x_5 = r$ and $x_6 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3.5r - t \\ -r \\ \frac{5}{3}t \\ -3r - t \\ r \\ t \end{bmatrix}$$

1.2.**13** The system reduces to $\begin{bmatrix} x & & - & z & = & 0 \\ & y & + & 2z & = & 0 \\ & & & 0 & = & 1 \end{bmatrix}$.

There are no solutions.

1.2.**14** The system reduces to $\begin{bmatrix} x & + & 2y & & = & -2 \\ & & & z & = & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} x & = -2-2y \\ z & = 2 \end{bmatrix}$.

Let $y = t$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2-2t \\ t \\ 2 \end{bmatrix}$$

1.2.**15** The system reduces to $\begin{bmatrix} x & & & = & 4 \\ & y & & = & 2 \\ & & z & = & 1 \end{bmatrix}$.

1.2.**16** The system reduces to $\begin{bmatrix} x_1 + 2x_2 + 3x_3 & & +5x_5 & = & 6 \\ & x_4 & +2x_5 & = & 7 \end{bmatrix} \longrightarrow$

$\begin{bmatrix} x_1 & = 6 - 2x_2 - 3x_3 - 5x_5 \\ x_4 & = 7 - 2x_5 \end{bmatrix}$.

Let $x_2 = r,\ x_3 = s$, and $x_5 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2r - 3s - 5t \\ r \\ s \\ 7 - 2t \\ t \end{bmatrix}$$

1.2.**17** The system reduces to $\begin{bmatrix} x_1 & & & & & = & -\frac{8221}{4340} \\ & x_2 & & & & = & \frac{8591}{8680} \\ & & x_3 & & & = & \frac{4695}{434} \\ & & & x_4 & & = & -\frac{459}{434} \\ & & & & x_5 & = & \frac{699}{434} \end{bmatrix}$.

1.2.**18** a No, since the third column contains two leading ones.

b Yes

c No, since the third row contains a leading one, but the second row does not.

d Yes

1.2.**19** $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

1.2.**20** $a = 1$ (by property a. on page 16),
$c = 0$ (by property b. on page 16), and
$e = 0$ (by property c. on page 16).
For the values of b and d we have two possibilities: $b = 0, d = 1$ or b arbitrary, $d = 0$.

1.2.**21** $e = 0$ (by property c. on page 16),
$c = 1$ (c must be a leading 1),
$d = 0$ (by property b. on page 16),
$b = 0$ (by property b. on page 16), and

a is arbitrary.

1.2.**22** Four, namely $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (k is an arbitrary constant.)

1.2.**23** Four, namely $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & k \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (k is an arbitrary constant.)

1.2.**24** Seven, namely $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & d \\ 0 & 1 & e \end{bmatrix}$, $\begin{bmatrix} 1 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Here, $a, b, \ldots, f$ are arbitrary constants.

1.2.**25** The conditions a, b, and c for the reduced row-echelon form correspond to the properties P1, P2, and P3 given on Page 13. The Gauss-Jordan algorithm, summarized on Page 15, guarantees that those properties are satisfied.

1.2.**26** Yes; each elementary row operation is reversible, that is, it can be "undone." For example, the operation of row swapping can be undone by swapping the same rows again. The operation of dividing a row by a scalar can be reversed by multiplying the same row by the same scalar.

1.2.**27** Yes; if A is transformed into B by a sequence of elementary row operations, then we can recover A from B by applying the inverse operations in the reversed order.

1.2.**28** Yes, by Exercise 27, since rref(A) is obtained from A by a sequence of elementary row operations.

1.2.**29** No; whatever elementary row operations you apply to $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, you cannot make the last column equal to zero.

1.2.**30** Suppose $(c_1, c_2, \ldots, c_n)$ is a solution of the system $\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \end{bmatrix}$.

To keep the notation simple, suppose we add k times the first equation to the second; then the second equation of the new system will be $(a_{21} + ka_{11})x_1 + \cdots + (a_{2n} + ka_{1n})x_n = b_2 + kb_1$.

We have to verify that $(c_1, c_2, \ldots, c_n)$ is a solution of this new equation. Indeed, $(a_{21} + ka_{11})c_1 + \cdots + (a_{2n} + ka_{1n})c_n = a_{21}c_1 + \cdots + a_{2n}c_n + k(a_{11}c_1 + \cdots + a_{1n}c_n) = b_2 + kb_1$.

We have shown that any solution of the "old" system is also a solution of the "new." To see that, conversely, any solution of the new system is also a solution of the old system, note that elementary row operations are reversible (compare with Exercise 26); we can obtain the old system by subtracting k times the first equation from the second equation of the new system.

1.2.**31** Since the number of oxygen atoms remains constant, we must have $2a + b = 2c + 3d$.

Considering hydrogen and nitrogen as well, we obtain the system $\begin{bmatrix} 2a & + & b & = & 2c & + & 3d \\ & & 2b & = & c & + & d \\ a & & & = & c & + & d \end{bmatrix}$ or

$\begin{bmatrix} 2a & + & b & - & 2c & - & 3d & = & 0 \\ & & 2b & - & c & - & d & = & 0 \\ a & & & - & c & - & d & = & 0 \end{bmatrix}$, which reduces to $\begin{bmatrix} a & & & - & 2d & = & 0 \\ & b & & - & d & = & 0 \\ & & c & - & d & = & 0 \end{bmatrix}$.

The solutions are $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix}$.

To get the smallest positive integers, we set $t = 1$:

$2NO_2 + H_2O \longrightarrow HNO_2 + HNO_3$

1.2.**32** Plugging the points into $f(t)$, we obtain the system

$$\begin{bmatrix} a & & & & & & & = & 1 \\ a & + & b & + & c & + & d & = & 0 \\ a & - & b & + & c & - & d & = & 0 \\ a & + & 2b & + & 4c & + & 8d & = & -15 \end{bmatrix}$$

with unique solution $a = 1,\ b = 2,\ c = -1$, and $d = -2$, so that $f(t) = 1 + 2t - t^2 - 2t^3$. (See Figure 1.8.)

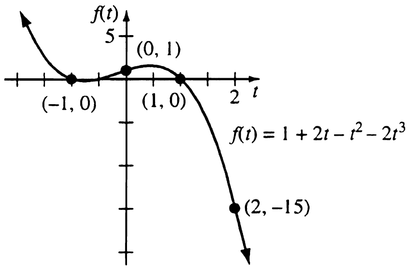


Figure 1.8: for Problem 1.2.32.

1.2.**33** Let $f(t) = a + bt + ct^2 + dt^3 + et^4$. Substituting the points in, we get

$$\begin{bmatrix} a & + & b & + & c & + & d & + & e & = & 1 \\ a & + & 2b & + & 4c & + & 8d & + & 16e & = & -1 \\ a & + & 3b & + & 9c & + & 27d & + & 81e & = & -59 \\ a & - & b & + & c & - & d & + & e & = & 5 \\ a & - & 2b & + & 4c & - & 8d & + & 16e & = & -29 \end{bmatrix}$$

This system has the unique solution $a = 1,\ b = -5,\ c = 4,\ d = 3$, and $e = -2$, so that $f(t) = 1 - 5t + 4t^2 + 3t^3 - 2t^4$. (See Figure 1.9.)

1.2.**34** The requirement $f_i'(a_i) = f_{i+1}'(a_i)$ and $f_i''(a_i) = f_{i+1}''(a_i)$ ensure that at each junction two different cubics fit "into" one another in a "smooth" way, since they must have the same slope and be equally curved. The requirement that $f_1'(a_0) = f_n'(a_n) = 0$ ensures that the track is horizontal at the beginning and at the end. How

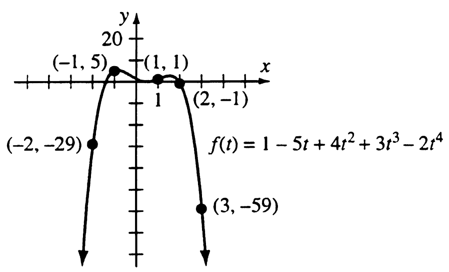


Figure 1.9: for Problem 1.2.33.

many unknowns are there? There are n pieces to be fit, and each one is a cubic of the form $f(t) = p + qt + rt^2 + st^3$, with p, q, r, and s to be determined; therefore, there are $4n$ unknowns. How many equations are there?

$f_i(a_i) = b_i$	for $i = 1, 2, \ldots, n$	gives n equations
$f_i(a_{i-1}) = b_{i-1}$	for $i = 1, 2, \ldots, n$	gives n equations
$f'_i(a_i) = f'_{i+1}(a_i)$	for $i = 1, 2, \ldots, n-1$	gives $n-1$ equations
$f''_i(a_i) = f''_{i+1}(a_i)$	for $i = 1, 2, \ldots, n-1$	gives $n-1$ equations
$f'_1(a_0) = 0, f'_n(a_n) = 0$		gives 2 equations

Altogether, we have $4n$ equations; convince yourself that all these equations are linear.

1.2.**35** Let $f(t) = a + bt + ct^2 + dt^3$, so that $f'(t) = b + 2ct + 3dt^2$.

Substituting the given points into $f(t)$ and $f'(t)$ we obtain the system

$$\begin{bmatrix} a & + & b & + & c & + & d & = & 1 \\ a & + & 2b & + & 4c & + & 8d & = & 5 \\ & & b & + & 2c & + & 3d & = & 2 \\ & & b & + & 4c & + & 12d & = & 9 \end{bmatrix}$$

This system has the unique solution $a = -5, b = 13, c = -10$, and $d = 3$, so that $f(t) = -5 + 13t - 10t^2 + 3t^3$. (See Figure 1.10.)

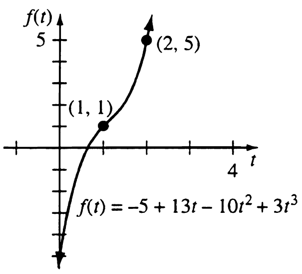


Figure 1.10: for Problem 1.2.35.

1.2.**36** We want all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = x + 3y - z = 0$. The endpoints of these vectors form a plane.

These vectors are of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3r+t \\ r \\ t \end{bmatrix}$, where r and t are arbitrary real numbers.

1.2.**37** We need to solve the system $\begin{bmatrix} x_1+x_2+x_3+x_4 & =0 \\ x_1+2x_2+3x_3+4x_4 & =0 \\ x_1+9x_2+9x_3+7x_4 & =0 \end{bmatrix}$,

which reduces to $\begin{bmatrix} x_1 & & & + & 0.25x_4 & = & 0 \\ & x_2 & & - & 1.5x_4 & = & 0 \\ & & x_3 & + & 2.25x_4 & = & 0 \end{bmatrix}$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -0.25t \\ 1.5t \\ -2.25t \\ t \end{bmatrix}$, where t is an arbitrary real number.

1.2.**38** Writing the equation $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ in terms of its components, we obtain the system

$$\begin{bmatrix} x_1+2x_2+4x_3 & =-8 \\ 4x_1+5x_2+6x_3 & =-1 \\ 7x_1+8x_2+9x_3 & = 9 \\ 5x_1+3x_2+x_3 & = 15 \end{bmatrix}$$

The system has the unique solution $x_1 = 2,\ x_2 = 3$, and $x_3 = -4$.

1.2.**39** Compare with the solution of Exercise 1.1.25.

The diagram tells us that $\begin{bmatrix} x_1 & =0.2x_2+0.3x_3+320 \\ x_2 & =0.1x_1+0.4x_3+90 \\ x_3 & =0.2x_1+0.5x_2+150 \end{bmatrix}$ or $\begin{bmatrix} x_1-0.2x_2-0.3x_3 & =320 \\ -0.1x_1+x_2-0.4x_3 & = 90 \\ -0.2x_1-0.5x_2+x_3 & =150 \end{bmatrix}$.

This system has the unique solution $x_1 = 500,\ x_2 = 300$, and $x_3 = 400$.

1.2.**40** a $\vec{v}_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0.2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0.2 \\ 0 \\ 0.5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 320 \\ 90 \\ 150 \end{bmatrix}$

b Recall that x_j is the output of industry I_j, and the ith component a_{ij} of $\vec{v}_j$ is the demand of Industry I_j on industry I_j for each dollar of output of industry I_j.

Therefore, the product $x_j a_{ij}$ (that is, the ith component of $x_j\vec{v}_j$), represents the total demand of industry I_j on Industry I_i (in dollars).

c $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n + \vec{b}$ is the vector whose ith component represents the total demand on industry I_i (consumer demand and interindustry demand combined).

d The ith component of the equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}$ expresses the requirement that the output x_i of industry I_i equal the total demand on that industry.

1.2.**41** a These components are zero because neither manufacturing not the energy sector directly require agricultural products.

b We have to solve the system $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + \vec{b} = \vec{x}$ or

$$\begin{bmatrix} 0.707x_1 & & & & & = & 13.2 \\ -0.014x_1 & + & 0.793x_2 & - & 0.017x_3 & = & 17.6 \\ -0.044x_1 & + & 0.01x_2 & + & 0.784x_3 & = & 1.8 \end{bmatrix}$$

The unique solution is approximately $x_1 = 18.67,\ x_2 = 22.60$, and $x_3 = 3.63$.

1.2.**42** We want to find m_1, m_2, m_3 such that $m_1 + m_2 + m_3 = 1$ and

$\frac{1}{1}\left(m_1\begin{bmatrix}1\\2\end{bmatrix} + m_2\begin{bmatrix}2\\3\end{bmatrix} + m_3\begin{bmatrix}4\\1\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix}$, that is, we have to solve the system

$$\begin{bmatrix} m_1 + m_2 + m_3 & = 1 \\ m_1 + 2m_2 + 4m_3 & = 2 \\ 2m_1 + 3m_2 + m_3 & = 2 \end{bmatrix}.$$

The unique solution is $m_1 = \frac{1}{2},\ m_2 = \frac{1}{4}$, and $m_3 = \frac{1}{4}$.

We will put $\frac{1}{2}$ kg at the point $\begin{bmatrix}1\\2\end{bmatrix}$ and $\frac{1}{4}$ kg at each of the two other vertices.

1.2.**43** We know that $m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{w}_1 + m_2\vec{w}_2$ or $m_1(\vec{v}_1 - \vec{w}_1) + m_2(\vec{v}_2 - \vec{w}_2) = \vec{0}$
or $\begin{bmatrix} -3m_1 + 2m_2 & = 0 \\ -6m_1 + 4m_2 & = 0 \\ -3m_1 + 2m_2 & = 0 \end{bmatrix}$.

We can conclude that $m_1 = \frac{2}{3}m_2$.

1.2.**44** Let x_1, x_2, x_3, and x_4 be the traffic volume at the four locations indicated in Figure 1.11.

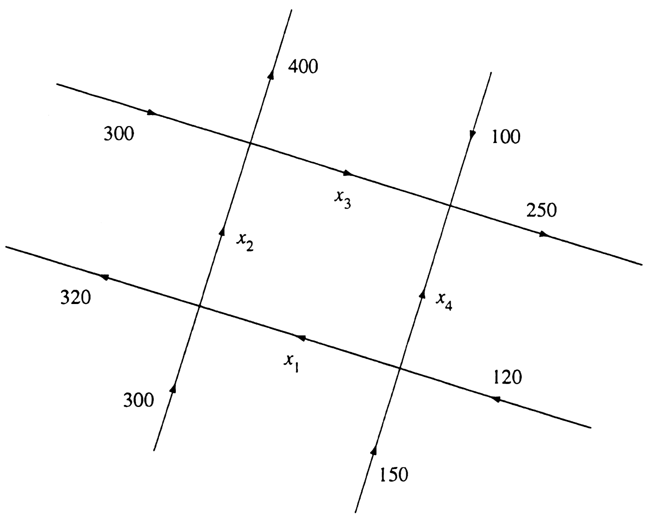


Figure 1.11: for Problem 1.2.44.

We are told that the number of cars coming into each intersection is the same as the number of cars coming out:

$$\begin{bmatrix} x_1+300 & =320+x_2 \\ x_2+300 & =400+x_3 \\ x_3+x_4+100 & =250 \\ 150+120 & =x_1+x_4 \end{bmatrix} \text{ or } \begin{bmatrix} x_1 & - & x_2 & & & & & = & 20 \\ & & x_2 & - & x_3 & & & = & 100 \\ & & & & x_3 & + & x_4 & = & 150 \\ x_1 & & & & & + & x_4 & = & 270 \end{bmatrix}$$

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 270-t \\ 250-t \\ 150-t \\ t \end{bmatrix}$.

Since the x_i must be positive integers (or zero), t must be an integer with $0 \le t \le 150$.

The lowest possible values are $x_1 = 120$, $x_2 = 100$, $x_3 = 0$, and $x_4 = 0$, while the highest possible values are $x_1 = 270$, $x_2 = 250$, $x_3 = 150$, and $x_4 = 150$.

1.2.**45** Plugging the data into the function $S(t)$ we obtain the system

$$\begin{bmatrix} a + b\cos\left(\frac{2\pi\cdot 47}{365}\right) + c\sin\left(\frac{2\pi\cdot 47}{365}\right) & = 11.5 \\ a + b\cos\left(\frac{2\pi\cdot 74}{365}\right) + c\sin\left(\frac{2\pi\cdot 74}{365}\right) & = 12 \\ a + b\cos\left(\frac{2\pi\cdot 273}{365}\right) + c\sin\left(\frac{2\pi\cdot 273}{365}\right) & = 12 \end{bmatrix}$$

The unique solution is approximately $a = 12.17$, $b = -1.15$, and $c = 0.18$, so that

$S(t) = 12.17 - 1.15\cos\left(\frac{2\pi t}{365}\right) + 0.18\sin\left(\frac{2\pi t}{365}\right)$.

The longest day is about 13.3 hours. (See Figure 1.12.)

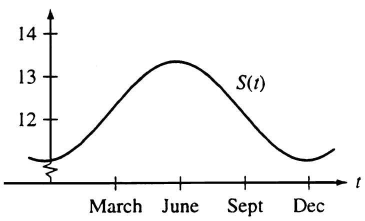


Figure 1.12: for Problem 1.2.45.

1.2.**46** Kyle first must solve the following system: $\begin{bmatrix} x_1 & +x_2 & +x_3 & =24 \\ 3x_1 & +2x_2 & +\frac{1}{2}x_3 & =24 \end{bmatrix}$.

This system reduces to $\begin{bmatrix} x_1 & & -1.5x_3 & =-24 \\ & x_2 & +2.5x_3 & =48 \end{bmatrix}$.

Thus, our solutions will be of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.5x_3 - 24 \\ -2.5x_3 + 48 \\ x_3 \end{bmatrix}$. Since all of our values must be non-negative integers (and x_3 must be even), we find the following solutions for $\begin{bmatrix} lilies \\ roses \\ daisies \end{bmatrix}$: $\begin{bmatrix} 0 \\ 8 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 18 \end{bmatrix}$. Since Olivia loves lilies, Kyle spends his 24 dollars on 3 lilies, 3 roses and 18 daisies.

1.2.**47** a When $k \neq 1$ and $k \neq 2$, we can see that this will continue to reduce to a consistent system with a unique solution.

b When $k = 1$, our bottom row reveals the inconsistency $0 = 2$.

c When $k = 2$, the second row and third row both represent the equation $z = 2$, meaning that the third row will be replaced with the equation $0 = 0$ during further reduction. This reveals that we will have an infinite number of solutions.

1.2.**48** a We reduce our matrix in the following steps: $\begin{bmatrix} 0 & 1 & 2k & \vdots & 0 \\ 1 & 2 & 6 & \vdots & 2 \\ k & 0 & 2 & \vdots & 1 \end{bmatrix} \begin{matrix} swap: \\ I \leftrightarrow II \end{matrix} \rightarrow$

$$\begin{bmatrix} 1 & 2 & 6 & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ k & 0 & 2 & \vdots & 1 \end{bmatrix} \begin{matrix} \\ \\ -k(I) \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 6 & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & -2k & 2-6k & \vdots & 1-2k \end{bmatrix} \begin{matrix} -2(II) \\ \\ +2k(II) \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 6-4k & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & 0 & 2-6k+4k^2 & \vdots & 1-2k \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 6-4k & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & 0 & 2(2k-1)(k-1) & \vdots & -(2k-1) \end{bmatrix}.$$

We see that there will be a unique solution when the $2(2k-1)(k-1)$ term is not equal to zero, when $2k - 1 \neq 0$ and $k - 1 \neq 0$, or $k \neq \frac{1}{2}$ and $k \neq 1$.

b We will have no solutions when the term $2(2k-1)(k-1)$ is equal to zero, but the term $-(2k-1)$ is not. This occurs only when $k = 1$.

c We will have infinitely many solutions when the last row represents the equation $0 = 0$. This occurs when $2k - 1 = 0$, or $k = \frac{1}{2}$.

1.2.**49** a So $-\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3 = 0$ and $-\frac{1}{2}x_2 + x_3 - \frac{1}{2}x_4 = 0$.

After reduction of the system, we find that our solutions are all of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

b Yes, from our solution in part (a), if we plug in 1 for x_1 and 13 for x_4, we obtain $3t - 2s = 1$ and $s = 13$, which leads to $t = 9$, and $x_2 = 5, x_3 = 9$.

So we have the solution: $x_1 = 1$, $x_2 = 5$, $x_3 = 9$ and $x_4 = 13$, which is an arithmetic progression.

1.2.**50** It is required that $x_k = \frac{1}{2}(x_{k-1} + x_{k+1})$, or $2x_k = x_{k-1} + x_{k+1}$, or $x_k - x_{k-1} = x_{k+1} - x_k$. This means that the difference of any two consecutive terms must be the same; we are looking at the finite arithmetic sequences. Thus the solutions are of the form $(x_1, x_2, x_3, \ldots, x_n) = (t, t+r, t+2r, \ldots, t+(n-1)r)$, where t and r are arbitrary constants.

1.2.**51** We begin by solving the system. Our augmented matrix begins as: $\begin{bmatrix} 2 & 1 & 0 & \vdots & C \\ 0 & 3 & 1 & \vdots & C \\ 1 & 0 & 4 & \vdots & C \end{bmatrix}$

and is reduced to $\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{9}{25}C \\ 0 & 1 & 0 & \vdots & \frac{7}{25}C \\ 0 & 0 & 1 & \vdots & \frac{4}{25}C \end{bmatrix}$. In order for x, y and z to be integers, C must be a multiple of 25. We want the smallest positive choice, so $C = 25$.

1.2.**52** $f(t) = a + bt + ct^2 + dt^3$ and we learn that $f(0) = a = 3$, $f(1) = a + b + c + d = 2$, $f(2) = a + 2b + 4c + 8d = 0$. Also,

$$\int_0^2 f(t)dt = at + \frac{1}{2}bt^2 + \frac{1}{3}ct^3 + \frac{1}{4}dt^4|_0^2 = 2a + 2b + \frac{8}{3}c + 4d = 4.$$

Now, we set up our matrix, $\begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 3 \\ 1 & 1 & 1 & 1 & \vdots & 2 \\ 1 & 2 & 4 & 8 & \vdots & 0 \\ 2 & 2 & \frac{8}{3} & 4 & \vdots & 4 \end{bmatrix}$. However, when we reduce this, the last line becomes $0 = 1$, meaning that the system is inconsistent.

In introductory calculus you may have seen the approximation formula:

$$\int_a^b f(t)dt \approx \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)),$$

the simplest form of Simpson's Rule. For polynomials $f(t)$ of degree ≤ 3, Simpson's Rule gives the exact value of the integral. Thus, for the $f(t)$ in our problem,

$$\int_0^2 f(t)dt = \frac{2}{6}(f(0) + 4f(1) + f(2)) = \frac{1}{3}(3 + 8 + 0) = \frac{11}{3}.$$

Thus it is impossible to find such a cubic with

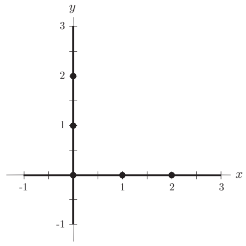
$$\int_0^2 f(t)dt = 4,$$

as required.

1.2.**53** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + 2c_3 + 4c_6 &= 0 \end{aligned}$$

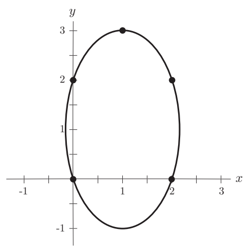
It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, 0, 0, c_5, 0)$. This is the conic $xy = 0$.



1.2.**54** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + 2c_3 + 4c_6 &= 0 \\ c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\ c_1 + c_2 + 3c_3 + c_4 + 3c_5 + 9c_6 &= 0 \end{aligned}$$

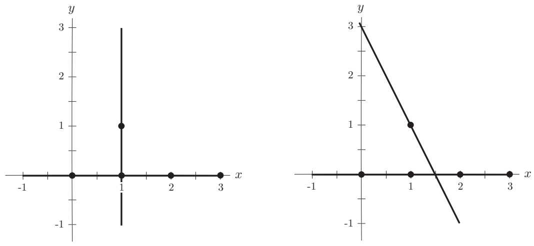
It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, -6c_6, -2c_6, 3c_6, 0, c_6)$. This is the conic $-6x - 2y + 3x^2 + y^2 = 0$ or $3(x-1)^2 + (y-1)^2 = 4$, the ellipse centered at $(1, 1)$ with semiaxis $2/\sqrt{3}$ and 2.



1.2.**55** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + 3c_2 + 9c_4 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \end{aligned}$$

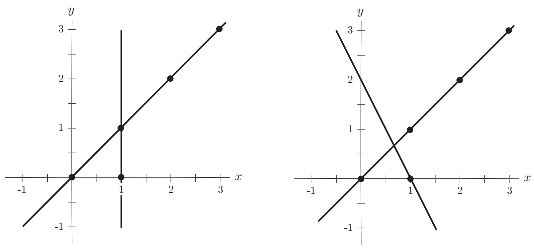
This system has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, -c_5 - c_6, 0, c_5, c_6)$. Setting $c_5 = a$ and $c_6 = b$, we get the family of conics $a(xy - y) + b(y^2 - y) = ay(x - 1) + by(y - 1) = 0$ where $a \neq 0$ or $b \neq 0$. Each such conic is the union of the x axis with some line through the point $(1, 1)$. Two sample solutions are shown in the accompanying figures.



1.2.**56** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \\ c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\ c_1 + 3c_2 + 3c_3 + 9c_4 + 9c_5 + 9c_6 &= 0 \\ c_1 + c_2 + c_4 &= 0 \end{aligned}$$

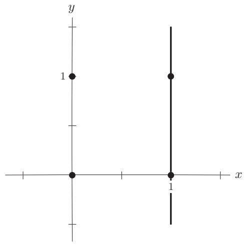
It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, c_5 + c_6, -c_5 - c_6, -c_5 - c_6, c_5, c_6)$. Setting $c_5 = a$ and $c_6 = b$, we get the family of conics $a(x - y + xy - x^2) + b(x - y - x^2 + y^2) = a(x - y)(1 - x) + b(x - y)(1 - x - y) = 0$ where $a \neq 0$ or $b \neq 0$. Each such conic is the union of the line $y = x$ with some line through the point $(1, 0)$. Two sample solutions are shown in the accompanying figures.

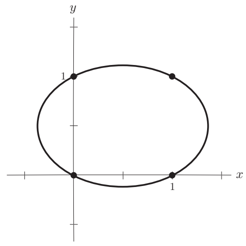


1.2.**57** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \end{aligned}$$

It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, -c_4, -c_6, c_4, 0, c_6)$. Setting $c_4 = a$ and $c_6 = b$, we get the family of conics $a(x^2 - x) + b(y^2 - y) = ax(x - 1) + by(y - 1) = 0$, where $a \neq 0$ or $b \neq 0$. Two sample solutions are shown in the accompanying figures.

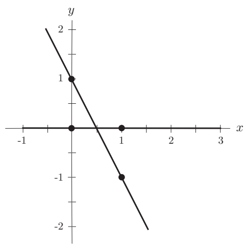


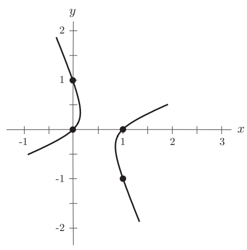


1.2.**58** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + c_2 - c_3 + c_4 - c_5 + c_6 &= 0 \end{aligned}$$

It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, -c_4, -c_6, c_4, 2c_6, c_6)$. Setting $c_4 = a$ and $c_6 = b$, we get the family of conics $a(x^2 - x) + b(y^2 + 2xy - y) = ax(x-1) + by(y + 2x - 1) = 0$, where $a \neq 0$ or $b \neq 0$. Two sample solutions are shown in the accompanying figures.

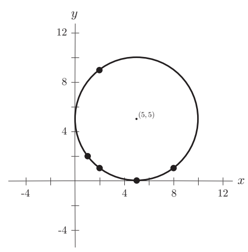




1.2.**59** The system of linear equations is

$$\begin{aligned} c_1 + 5c_2 + 25c_4 &= 0 \\ c_1 + c_2 + 2c_3 + c_4 + 2c_5 + 4c_6 &= 0 \\ c_1 + 2c_2 + c_3 + 4c_4 + 2c_5 + c_6 &= 0 \\ c_1 + 8c_2 + c_3 + 64c_4 + 8c_5 + c_6 &= 0 \\ c_1 + 2c_2 + 9c_3 + 4c_4 + 18c_5 + 81c_6 &= 0 \end{aligned}$$

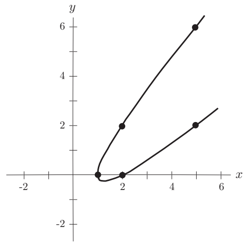
It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (25c_6, -10c_6, -10c_6, c_6, 0, c_6)$. This is the conic $25 - 10x - 10y + x^2 + y^2 = 0$, a circle of radius 5 centered at $(5, 5)$.



1.2.**60** The system of linear equations is

$$\begin{aligned} c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\ c_1 + 5c_2 + 2c_3 + 25c_4 + 10c_5 + 4c_6 &= 0 \\ c_1 + 5c_2 + 6c_3 + 25c_4 + 30c_5 + 36c_6 &= 0 \end{aligned}$$

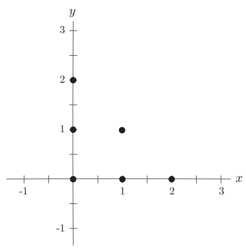
It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (2c_6, -3c_6, 2c_6, c_6, -2c_6, c_6)$. This is the conic $2-3x+2y+x^2-2xy+y^2 = 0$.



1.2.**61** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + 2c_3 + 4c_6 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \end{aligned}$$

It has the only solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, 0, 0, 0, 0)$. There is no conic which goes through these points. Alternatively, note that the only conic through the first five points is $xy = 0$, according to Exercise 51. But that conic fails to run through the point $(1, 1)$, so that there is no conic through all six points.

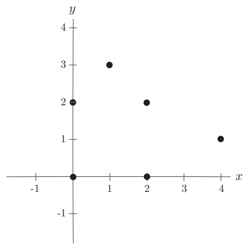


1.2.**62** The system of linear equations is

$$c_1 = 0$$

$$\begin{aligned} c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + 2c_3 + 4c_6 &= 0 \\ c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\ c_1 + c_2 + 3c_3 + c_4 + 3c_5 + 9c_6 &= 0 \\ c_1 + 4c_2 + c_3 + 16c_4 + 4c_5 + c_6 &= 0 \end{aligned}$$

It has the solution $(c_1, c_2, c_3, c_4, c_5, c_6) = (0,0,0,0,0,0)$. There is no conic which goes through these points.



1.2.**63** Let x_1 be the cost of the environmental statistics book, x_2 be the cost of the set theory text and x_3 be the cost of the educational psychology book. Then, from the problem, we deduce the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 178 \\ 2 & 1 & 1 & 319 \\ 0 & 1 & 1 & 147 \end{array}\right]$.

We can reduce this matrix to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 86 \\ 0 & 1 & 0 & 92 \\ 0 & 0 & 1 & 55 \end{array}\right]$, revealing that $x_1 = 86$, $x_2 = 92$ and $x_3 = 55$. Thus, the environmental statistics book costs \$ 86, the set theory book costs \$ 92 and the educational psychology book is only priced at \$ 55.

1.2.**64** Let our vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ represent the numbers of the books $\begin{bmatrix} grammar \\ Werther \\ LinearAlg. \end{bmatrix}$. Then we can set up the matrix $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 64 \\ 1 & 0 & 1 & 98 \\ 0 & 1 & 1 & 76 \end{array}\right]$. This system yields one solution, $\begin{bmatrix} 43 \\ 21 \\ 55 \end{bmatrix}$, meaning that the grammar book costs 43 Euro, the novel costs 21 Euro, and the linear algebra text costs 55 Euro.

1.2.**65** The difficult part of this problem lies in setting up a system from which we can derive our matrix. We will define x_1 to be the number of "liberal" students at the beginning of the class, and x_2 to be the number of "conservative" students at the beginning. Thus, since there are 260 students in total, $x_1 + x_2 = 260$. We need one more equation involving x_1 and x_2 in order to set up a useful system. Since we know that the number of conservative students at the end of the semester is equal to the number of liberal student initially, we obtain the equation $\frac{3}{10}x_1 + \frac{6}{10}x_2 = x_1$, or $-\frac{7}{10}x_1 + \frac{6}{10}x_2 = 0$.

We then use $\left[\begin{array}{cc|c} 1 & 1 & 260 \\ -\frac{7}{10} & \frac{6}{10} & 0 \end{array}\right]$ to obtain $\left[\begin{array}{cc|c} 1 & 0 & 120 \\ 0 & 1 & 140 \end{array}\right]$.

Thus, there are initially 120 liberal students, and 140 conservative students. Since the number of liberal students initially is the same as the number of conservative students in the end, the class ends with 120 conservative students and 140 liberal students.

1.2.**66** Let x_1 and x_2 be the initial number of students in Sections A and B, respectively. Then, since there are 55 students total, $x_1 + x_2 = 55$. Also, interpreting the change of students from the perspective of Section B, we gain $0.2x_1$, lose $0.3x_2$, and in the process, lose 4 students. Thus, $0.2x_1 - 0.3x_2 = -4.0$. Our matrix becomes $\begin{bmatrix} 1 & 1 & \vdots & 55 \\ .2 & -.3 & \vdots & -4 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & \vdots & 25 \\ 0 & 1 & \vdots & 30 \end{bmatrix}$. This reveals that there are initially 25 students in Section A and 30 students in Section B.

1.2.**67** We are told that five cows and two sheep cost ten liang, and two cows and five sheep cost eight liang of silver. So, we let C be the cost of a cow, and S be the cost of a sheep. From this we derive $\begin{bmatrix} 5C & +2S & = 10 \\ 2C & +5S & = 8 \end{bmatrix}$.

This reduces to $\begin{bmatrix} C & & = \frac{34}{21} \\ & S & = \frac{20}{21} \end{bmatrix}$ which gives the prices: $\frac{34}{21}$ liang silver for a cow, and $\frac{20}{21}$ liang silver for a sheep.

1.2.**68** Letting x_1, x_2, and x_3 be the prize, in coins, of cows, sheep and pigs, respectively, we can represent the system in a matrix: $\begin{bmatrix} 2 & 5 & -13 & \vdots & 1000 \\ 3 & -9 & 3 & \vdots & 0 \\ -5 & 6 & 8 & \vdots & -600 \end{bmatrix}$. We reduce this matrix to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 1200 \\ 0 & 1 & 0 & \vdots & 500 \\ 0 & 0 & 1 & \vdots & 300 \end{bmatrix}$. The prize of a cow, a sheep, and a pig is 1200, 500 and 300 coins, respectively.

1.2.**69** The second measurement in the problem tells us that 4 sparrows and 1 swallow weigh as much as 1 sparrow and 5 swallows. We will immediately interpret this as 3 sparrows weighing the same as 4 swallows. The other measurement we use is that all the birds together weigh 16 liang. Setting x_1 to be the weight of a sparrow, and x_2 to be the weight of a swallow, we find the augmented matrix $\begin{bmatrix} 3 & -4 & \vdots & 0 \\ 5 & 6 & \vdots & 16 \end{bmatrix}$ representing these two equations.

We reduce this to $\begin{bmatrix} 1 & 0 & \vdots & \frac{32}{19} \\ 0 & 1 & \vdots & \frac{24}{19} \end{bmatrix}$, meaning that each sparrow weighs $\frac{32}{19}$ liang, and each swallow weighs $\frac{24}{19}$ liang.

1.2.**70** This problem gives us three different combinations of horses that can pull exactly 40 *dan* up a hill. We condense the statements to fit our needs, saying that, One military horse and one ordinary horse can pull 40 *dan*, two ordinary and one weak horse can pull 40 *dan* and one military and three weak horses can also pull 40 *dan*.

With this information, we set up our matrix: $\begin{bmatrix} 1 & 1 & 0 & \vdots & 40 \\ 0 & 2 & 1 & \vdots & 40 \\ 1 & 0 & 3 & \vdots & 40 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{40}{7} \\ 0 & 1 & 0 & \vdots & \frac{120}{7} \\ 0 & 0 & 1 & \vdots & \frac{40}{7} \end{bmatrix}$.

Thus, the military horses can pull $\frac{40}{7}$ *dan*, the ordinary horses can pull $\frac{120}{7}$ *dan* and the weak horses can pull $\frac{40}{7}$ *dan* each.

1.2.**71** Here, let W be the depth of the well.

Then our system becomes $\begin{bmatrix} 2A & +B & & & & -W & = & 0 \\ & 3B & +C & & & -W & = & 0 \\ & & 4C & +D & & -W & = & 0 \\ & & & 5D & +E & -W & = & 0 \\ A & & & & +6E & -W & = & 0 \end{bmatrix}$.

We transform this system into an augmented matrix, then perform a prolonged reduction to reveal
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{265}{721} & \vdots & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{191}{721} & \vdots & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{148}{721} & \vdots & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{129}{721} & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{76}{721} & \vdots & 0 \end{bmatrix}$. Thus, $A = \frac{265}{721}W$, $B = \frac{191}{721}W$, $C = \frac{148}{721}W$, $D = \frac{129}{721}W$ and $E = \frac{76}{721}W$.

If we choose 721 to be the depth of the well, then $A = 265$, $B = 191$, $C = 148$, $D = 129$ and $E = 76$.

1.2.**72** We let x_1, x_2 and x_3 be the numbers of roosters, hens and chicks respectively. Then, since we buy a total of a hundred birds, and spend a hundred coins on them, we find the equations $x_1 + x_2 + x_3 = 100$ and $5x_1 + 3x_2 + \frac{1}{3}x_3 = 100$.

We fit these into our matrix, $\begin{bmatrix} 1 & 1 & 1 & \vdots & 100 \\ 5 & 3 & \frac{1}{3} & \vdots & 100 \end{bmatrix}$,

which reduces to $\begin{bmatrix} 1 & 0 & -\frac{4}{3} & \vdots & -100 \\ 0 & 1 & \frac{7}{3} & \vdots & 200 \end{bmatrix}$.

So, $x_1 - \frac{4}{3}x_3 = -100$, and $x_2 + \frac{7}{3}x_3 = 200$. Now, we can write our solution vectors in terms of x_3: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 100 \\ -\frac{7}{3}x_3 + 200 \\ x_3 \end{bmatrix}$. Since all of our values must be non-negative, x_1 must be greater than or equal to zero, or $\frac{4}{3}x_3 - 100 \geq 0$, which means that $x_3 \geq 75$.

Also, x_3 must be greater than or equal to zero, meaning that $-\frac{7}{3}x_3 + 200 \geq 0$ or $x_3 \leq \frac{600}{7}$. Since x_3 must be an integer, this forces $x_3 \leq 85$. Thus, we are looking for solutions where $75 \leq x_3 \leq 85$. We notice, however, that x_1 and x_2 are only integers when x_3 is a multiple of 3. Thus, the possible values for x_3 are 75, 78, 81 and 84.

Now the possible solutions for $\begin{bmatrix} \textit{roosters} \\ \textit{hens} \\ \textit{chicks} \end{bmatrix}$ are $\begin{bmatrix} 0 \\ 25 \\ 75 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 18 \\ 78 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 11 \\ 81 \end{bmatrix}$, and $\begin{bmatrix} 12 \\ 4 \\ 84 \end{bmatrix}$.

1.2.**73** We let x_1, x_2, x_3 and x_4 be the numbers of pigeons, sarasabirds, swans and peacocks respectively. We first determine the cost of each bird. Each pigeon costs $\frac{3}{5}$ panas, each sarasabird costs $\frac{5}{7}$ panas, the swans cost $\frac{7}{9}$ panas apiece and each peacock costs 3 panas. We use these numbers to set up our system, but we must remember to make sure we are buying the proper amount of each to qualify for these deals when we find our solutions (for example, the number of sarasabirds we buy must be a multiple of 7).

Our matrix then is $\begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 100 \\ \frac{3}{5} & \frac{5}{7} & \frac{7}{9} & 3 & \vdots & 100 \end{bmatrix}$

which reduces to $\begin{bmatrix} 1 & 0 & -\frac{5}{9} & -20 & \vdots & -250 \\ 0 & 1 & \frac{14}{9} & 21 & \vdots & 350 \end{bmatrix}$.

Thus, $x_1 = \frac{5}{9}x_3 + 20x_4 - 250$ and $x_2 = -\frac{14}{9}x_3 - 21x_4 + 350$.

Then our solutions are of the form $\begin{bmatrix} \frac{5}{9}x_3 + 20x_4 - 250 \\ -\frac{14}{9}x_3 - 21x_4 + 350 \\ x_3 \\ x_4 \end{bmatrix}$.

We determine the possible solutions by choosing combinations of x_3 and x_4 of the correct multiples (9 for x_3, 3 for x_4) that give us non-negative integer solutions for x_1 and x_2. Thus it is required that $x_1 = \frac{5}{9}x_3 + 20x_4 - 250 \geq 0$ and $x_2 = -\frac{14}{9}x_3 - 21x_4 + 350 \geq 0$.

Solving for x_3 we find that $225 - \frac{27}{2}x_4 \geq x_3 \geq 450 - 36x_4$.

To find all the solutions, we can begin by letting $x_4 = 0$, and finding all corresponding values of x_3. Then we can increase x_4 in increments of 3, and find the corresponding x_3 values in each case, until we are through.

For $x_4 = 0$ we have the inequality $225 \geq x_3 \geq 450$, so that there aren't any solutions for x_3. Likewise, there are no feasible x_3 values for $x_4 = 3, 6$ and 9, since $450 - 36x_4$ exceeds 100 in these cases.

In the case of $x_4 = 12$ our inequality becomes $63 \geq x_3 \geq 18$, so that x_3 could be 18, 27, 36, 45, 54 or 63.

In the next case, $x_4 = 15$, we have $\frac{45}{2} \geq x_3 \geq -90$, so that the non-negative solutions are 0, 9 and 18.

If x_4 is 18 or more, then the term $225 - \frac{27}{2}x_4$ becomes negative, so that there are only negative solutions for x_3. (Recall that it is required that $225 - \frac{27}{2}x_4 \geq x_3$.)

We have found nine solutions. If we compute the corresponding values of

$x_1 = \frac{5}{9}x_3 + 20x_4 - 250$ and $x_2 = -\frac{14}{9}x_3 - 21x_4 + 350$, we end up with the following vectors for: $\begin{bmatrix} \text{number of pigeons} \\ \text{number of sarasabirds} \\ \text{number of swans} \\ \text{number of peacocks} \end{bmatrix}$

to be: $\begin{bmatrix} 0 \\ 70 \\ 18 \\ 12 \end{bmatrix}, \begin{bmatrix} 5 \\ 56 \\ 27 \\ 12 \end{bmatrix}, \begin{bmatrix} 10 \\ 42 \\ 36 \\ 12 \end{bmatrix}, \begin{bmatrix} 15 \\ 28 \\ 45 \\ 12 \end{bmatrix}, \begin{bmatrix} 20 \\ 14 \\ 54 \\ 12 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ 63 \\ 12 \end{bmatrix}, \begin{bmatrix} 50 \\ 35 \\ 0 \\ 15 \end{bmatrix}, \begin{bmatrix} 55 \\ 21 \\ 9 \\ 15 \end{bmatrix}, \begin{bmatrix} 60 \\ 7 \\ 18 \\ 15 \end{bmatrix}$.

1.2.**74** We follow the outline of Exercise 72 to find the matrix $\begin{bmatrix} 1 & 1 & 1 & \vdots & 100 \\ 4 & \frac{1}{5} & 1 & \vdots & 100 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & \frac{4}{19} & \vdots & \frac{400}{19} \\ 0 & 1 & \frac{15}{19} & \vdots & \frac{1500}{19} \end{bmatrix}$.

Thus, our solutions are of the form $\begin{bmatrix} \frac{400-4x_3}{19} \\ \frac{1500-15x_3}{19} \\ x_3 \end{bmatrix}$. We find that our solutions are bound by $0 \leq x_3 \leq 100$. However, since both $\frac{400-4x_3}{19} = 4\frac{100-x_3}{19}$ and $\frac{1500-15x_3}{19} = 15\frac{100-x_3}{19}$ must be non-negative integers, the quantity $\frac{100-x_3}{19}$ must be a non-negative integer, k, so that $x_3 = 100 - 19k$. The condition $x_3 \geq 0$ now leaves us with the possibilities $k = 0, 1, 2, 3, 4, 5$.

Thus, we find our solutions for $\begin{bmatrix} ducks \\ sparrows \\ roosters \end{bmatrix}$: $\begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$. $\begin{bmatrix} 4 \\ 15 \\ 81 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 30 \\ 62 \end{bmatrix}$, $\begin{bmatrix} 12 \\ 45 \\ 43 \end{bmatrix}$, $\begin{bmatrix} 16 \\ 60 \\ 24 \end{bmatrix}$ and $\begin{bmatrix} 20 \\ 75 \\ 5 \end{bmatrix}$.

1.2.**75** We let x_1 be the number of sheep, x_2 be the number of goats, and x_3 be the number of hogs. We can then use the two equations $\frac{1}{2}x_1 + \frac{4}{3}x_2 + \frac{7}{2}x_3 = 100$ and $x_1 + x_2 + x_3 = 100$ to generate the following augmented matrix:
$\begin{bmatrix} \frac{1}{2} & \frac{4}{3} & \frac{7}{2} \vdots & 100 \\ 1 & 1 & 1 \vdots & 100 \end{bmatrix}$

then reduce it to $\begin{bmatrix} 1 & 0 & -\frac{13}{5} \vdots & 40 \\ 0 & 1 & \frac{18}{5} \vdots & 60 \end{bmatrix}$.

With this, we see that our solutions will be of the form $\begin{bmatrix} 40 + \frac{13}{5}s \\ 60 - \frac{18}{5}s \\ s \end{bmatrix}$. Now all three components of this vector must be non-negative integers, meaning that s must be a non-negative multiple of 5 (that is, $s = 0, 5, 10, \ldots$) such that $60 - \frac{18}{5}s \geq 0$, or, $s \leq \frac{50}{3}$. This leaves the possible solutions $x_3 = s = 0, 5, 10$ and 15, and we can compute the corresponding values of $x_1 = 40 + \frac{13}{5}s$ and $x_2 = 60 - \frac{18}{5}s$ in each case.

So we find the following solutions: $\begin{bmatrix} 40 \\ 60 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 53 \\ 42 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 66 \\ 24 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 79 \\ 6 \\ 15 \end{bmatrix}$.

1.2.**76** This problem is similar in nature to Exercise 72, and we will follow that example, revealing the matrix: $\begin{bmatrix} 1 & 1 & 1 & \vdots & 100 \\ 3 & 2 & \frac{1}{2} & \vdots & 100 \end{bmatrix}$. We reduce this to $\begin{bmatrix} 1 & 0 & -\frac{3}{2} & \vdots & -100 \\ 0 & 1 & \frac{5}{2} & \vdots & 200 \end{bmatrix}$, which yields solutions of the form $\begin{bmatrix} \frac{3}{2}x_3 - 100 \\ -\frac{5}{2}x_3 + 200 \\ x_3 \end{bmatrix}$. Since all the values must be positive (there are at least one man, one woman and one child), we see that $66 < x_3 < 80$, and x_3 must be even. From this, we use x_3 to find our solutions:
$\begin{bmatrix} 2 \\ 30 \\ 68 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 25 \\ 70 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 20 \\ 72 \end{bmatrix}$, $\begin{bmatrix} 11 \\ 15 \\ 74 \end{bmatrix}$, $\begin{bmatrix} 14 \\ 10 \\ 76 \end{bmatrix}$ and $\begin{bmatrix} 17 \\ 5 \\ 78 \end{bmatrix}$.

1.2.**77** Rather than setting up a huge system, here we will reason this out logically. Since there are 30 barrels, each son will get 10 of them. If we use the content of a full barrel as our unit for wine, we see that each brother will get $\frac{15}{3} = 5$ barrel-fulls of wine. Thus, the ten barrels received by each son will, on average, be half full, meaning that for every full barrel a son receives, he also receives an empty one.

Now let x_1, x_2, and x_3 be the numbers of half-full barrels received by each of the three sons. The first son, receiving x_1 half-full barrels will also gain $10 - x_1$ other barrels, half of which must be full and half of which must be empty, each equal to the quantity $\frac{10-x_1}{2}$. Thus, x_1 must be even. The same works for x_2 and x_3. Since $x_1 + x_2 + x_3 = 10$, we have boiled down our problem to simply finding lists of three non-negative even numbers that add up to 10. We find our solutions by inspection:

$\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$,

$$\begin{bmatrix}2\\8\\0\end{bmatrix}, \begin{bmatrix}2\\6\\2\end{bmatrix}, \begin{bmatrix}2\\4\\4\end{bmatrix}, \begin{bmatrix}2\\2\\6\end{bmatrix}, \begin{bmatrix}2\\0\\8\end{bmatrix}, \begin{bmatrix}0\\10\\0\end{bmatrix}, \begin{bmatrix}0\\8\\2\end{bmatrix}, \begin{bmatrix}0\\6\\4\end{bmatrix}, \begin{bmatrix}0\\4\\6\end{bmatrix}, \begin{bmatrix}0\\2\\8\end{bmatrix} \text{ and } \begin{bmatrix}0\\0\\10\end{bmatrix}.$$

As we stated before, the number of full and empty barrels is dependent on the number of half-full barrels. Thus, each solution here translates into exactly one solution for the overall problem. Here we list those solutions, for $\begin{bmatrix}\text{first son}\\\text{second son}\\\text{third son}\end{bmatrix}$, using triples of the form (full barrels, half-full barrels, empty barrels) as our entries:

$$\begin{bmatrix}(0,10,0)\\(5,0,5)\\(5,0,5)\end{bmatrix}, \begin{bmatrix}(1,8,1)\\(4,2,4)\\(5,0,5)\end{bmatrix}, \begin{bmatrix}(1,8,1)\\(5,0,5)\\(4,2,4)\end{bmatrix}, \begin{bmatrix}(2,6,2)\\(3,4,3)\\(5,0,5)\end{bmatrix}, \begin{bmatrix}(2,6,2)\\(4,2,4)\\(4,2,4)\end{bmatrix}, \begin{bmatrix}(2,6,2)\\(5,0,5)\\(3,4,3)\end{bmatrix},$$

$$\begin{bmatrix}(3,4,3)\\(2,6,2)\\(5,0,5)\end{bmatrix}, \begin{bmatrix}(3,4,3)\\(3,4,3)\\(4,2,4)\end{bmatrix}, \begin{bmatrix}(3,4,3)\\(4,2,4)\\(3,4,3)\end{bmatrix}, \begin{bmatrix}(3,4,3)\\(5,0,5)\\(2,6,2)\end{bmatrix}, \begin{bmatrix}(4,2,4)\\(1,8,1)\\(5,0,5)\end{bmatrix}, \begin{bmatrix}(4,2,4)\\(2,6,2)\\(4,2,4)\end{bmatrix},$$

$$\begin{bmatrix}(4,2,4)\\(3,4,3)\\(3,4,3)\end{bmatrix}, \begin{bmatrix}(4,2,4)\\(4,2,4)\\(2,6,2)\end{bmatrix}, \begin{bmatrix}(4,2,4)\\(5,0,5)\\(1,8,1)\end{bmatrix}, \begin{bmatrix}(5,0,5)\\(0,10,0)\\(5,0,5)\end{bmatrix}, \begin{bmatrix}(5,0,5)\\(1,8,1)\\(4,2,4)\end{bmatrix}, \begin{bmatrix}(5,0,5)\\(2,6,2)\\(3,4,3)\end{bmatrix},$$

$$\begin{bmatrix}(5,0,5)\\(3,4,3)\\(2,6,2)\end{bmatrix}, \begin{bmatrix}(5,0,5)\\(4,2,4)\\(1,8,1)\end{bmatrix} \text{ and } \begin{bmatrix}(5,0,5)\\(5,0,5)\\(0,10,0)\end{bmatrix}.$$

1.2.**78** We let x_1 be the amount of gold in the crown, x_2 be the amount of bronze, x_3 be the amount of tin and x_4 be the amount of iron. Then, for example, since the first requirement in the problem is: "Let the gold and bronze together form two-thirds," we will interpret this as $x_1+x_2=\frac{2}{3}(60)$. We do this for all three requirements, and use the fact that all combined will be the total weight of the crown as our fourth. So we find the matrix $\begin{bmatrix}1&1&0&0&\vdots&\frac{2}{3}(60)\\1&0&1&0&\vdots&\frac{3}{4}(60)\\1&0&0&1&\vdots&\frac{3}{5}(60)\\1&1&1&1&\vdots&60\end{bmatrix}$, which has the solution $\begin{bmatrix}\textit{gold}\\\textit{bronze}\\\textit{tin}\\\textit{iron}\end{bmatrix}=\begin{bmatrix}30.5\\9.5\\14.5\\5.5\end{bmatrix}$.

1.2.**79** Let x_i be the number of coins the i^{th} merchant has. We interpret the statement of the first merchant, "If I keep the purse, I shall have twice as much money as the two of you together" as $x_1+60=2(x_2+x_3)$, or $-x_1+2x_2+2x_3=60$. We interpret the other statements in a similar fashion, translating this into the augmented matrix, $\begin{bmatrix}-1&2&2\vdots&60\\3&-1&3\vdots&60\\5&5&-1\vdots&60\end{bmatrix}$.

The reduced row echelon form of this matrix is $\begin{bmatrix}1&0&0\vdots&4\\0&1&0\vdots&12\\0&0&1\vdots&20\end{bmatrix}$. Thus we deduce that the first merchant has 4 coins, the second has 12, and the third is the richest, with 20 coins.

1.2.**80** For each of the three statements, we set up an equation of the form

(initial amount of grass) + (grass growth) = (grass consumed by cows), or

$(\#offields)x + (\#offields)(\#ofdays)y = (\#ofcows)(\#ofdays)z$.

For the first statement, this produces the equation $x + 2y = 6z$, or $x + 2y - 6z = 0$. Similarly, we obtain the equations $4x + 16y - 28z = 0$ and $2x + 10y - 15z = 0$ for the other two statements. From this information, we write the matrix $\begin{bmatrix} 1 & 2 & -6 & \vdots & 0 \\ 4 & 16 & -28 & \vdots & 0 \\ 2 & 10 & -15 & \vdots & 0 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & -5 & \vdots & 0 \\ 0 & 1 & -\frac{1}{2} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Thus our solutions are of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5t \\ \frac{1}{2}t \\ t \end{bmatrix}$, where t is an arbitrary positive real number.

Section 1.3

1.3.**1** a No solution, since the last row indicates $0 = 1$.

b The unique solution is $x = 5,\ y = 6$.

c Infinitely many solutions; the first variable can be chosen freely.

1.3.**2** The rank is 3 since each row contains a leading one.

1.3.**3** This matrix has rank 1 since its rref is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

1.3.**4** This matrix has rank 2 since its rref is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

1.3.**5** a $x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

b The solution of the system in part (a) is $x = 3,\ y = 2$. (See Figure 1.13.)

1.3.**6** No solution, since any linear combination $x\vec{v}_1 + y\vec{v}_2$ of $\vec{v}_1$ and $\vec{v}_2$ will be parallel to $\vec{v}_1$ and $\vec{v}_2$.

1.3.**7** A unique solution, since there is only one parallelogram with sides along $\vec{v}_1$ and $\vec{v}_2$ and one vertex at the tip of $\vec{v}_3$.

1.3.**8** Infinitely many solution. There are at least two obvious solutions. Write $\vec{v}_4$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ alone or as a linear combination of $\vec{v}_3$ and $\vec{v}_2$ alone. Therefore, this linear system has infinitely many solutions, by Theorem 1.3.1.

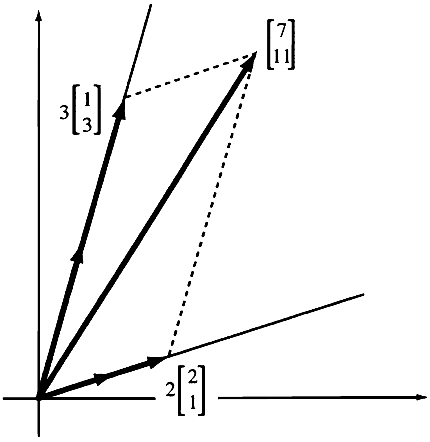


Figure 1.13: for Problem 1.3.5.

1.3.**9** $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\4\\9\end{bmatrix}$

1.3.**10** $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}=1\cdot1+2\cdot(-2)+3\cdot1=0$

1.3.**11** Undefined since the two vectors do not have the same number of components.

1.3.**12** $[1\ 2\ 3\ 4]\cdot\begin{bmatrix}5\\6\\7\\8\end{bmatrix}=1\cdot5+2\cdot6+3\cdot7+4\cdot8=70$

1.3.**13** $\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}7\\11\end{bmatrix}=7\begin{bmatrix}1\\3\end{bmatrix}+11\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}29\\65\end{bmatrix}$ or $\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}7\\11\end{bmatrix}=\begin{bmatrix}1\cdot7+2\cdot11\\3\cdot7+4\cdot11\end{bmatrix}=\begin{bmatrix}29\\65\end{bmatrix}$

1.3.**14** $\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix}\begin{bmatrix}-1\\2\\1\end{bmatrix}=-1\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}2\\3\end{bmatrix}+1\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$ or $\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix}\begin{bmatrix}-1\\2\\1\end{bmatrix}=$
$\begin{bmatrix}1\cdot(-1)+2\cdot2+3\cdot1\\2\cdot(-1)+3\cdot2+4\cdot1\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$

1.3.**15** $[1\ 2\ 3\ 4]\begin{bmatrix}5\\6\\7\\8\end{bmatrix}=5\cdot1+6\cdot2+7\cdot3+4\cdot8=70$ either way.

1.3.**16** $\begin{bmatrix}0&1\\3&2\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix}=\begin{bmatrix}0\cdot2+1\cdot(-3)\\3\cdot2+2\cdot(-3)\end{bmatrix}=\begin{bmatrix}-3\\0\end{bmatrix}$

1.3.**17** Undefined, since the matrix has three columns, but the vector has only two components.

1.3.**18** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$

1.3.**19** $\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

1.3.**20** a $\begin{bmatrix} 9 & 8 \\ 7 & 6 \\ 6 & 6 \end{bmatrix}$

b $\begin{bmatrix} 9 & -9 & 18 \\ 27 & 36 & 45 \end{bmatrix}$

1.3.**21** $\begin{bmatrix} 158 \\ 70 \\ 81 \\ 123 \end{bmatrix}$

1.3.**22** By Theorem 1.3.4, the rref is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1.3.**23** All variables are leading, that is, there is a leading one in each column of the rref: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

1.3.**24** By Theorem 1.3.4, rref $(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

1.3.**25** In this case, rref(A) has a row of zeros, so that rank(A) < 4; there will be a free variable. The system $A\vec{x} = \vec{c}$ could have infinitely many solutions (for example, when $\vec{c} = \vec{0}$) or no solutions (for example, when $\vec{c} = \vec{b}$), but it cannot have a unique solution, by Theorem 1.3.4.

1.3.**26** From Example 4c we know that rank(A) $= 3$, so that rref(A) $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Since all variables are leading, the system $A\vec{x} = \vec{c}$ cannot have infinitely many solutions, but it could have a unique solution (for example, if $\vec{c} = \vec{b}$) or no solutions at all (compare with Example 4d).

1.3.**27** By Theorem 1.3.4, rref $(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

1.3.**28** There must be a leading one in each column: rref $(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

1.3.**29** A is of the form $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

and $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5a \\ 3b \\ -9c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

So $a = \frac{2}{5}$, $b = 0$ and $c = -\frac{1}{9}$, and $A = \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}$

1.3.**30** We must satisfy the equation $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Thus, $5a + 3b - 9c = 2$, $5d + 3e - 9f = 0$, and $5g + 3h - 9i = 1$. One way to force our matrix to have rank 1 is to make all the entries in the second and third columns zero, meaning that $a = \frac{5}{2}$, $d = 0$, and $g = \frac{1}{5}$. Thus, one possible matrix is $\begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 \end{bmatrix}$.

1.3.**31** A is of the form $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

and $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5a + 3b - 9c \\ 3d - 9e \\ -9f \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Clearly, f must equal $-\frac{1}{9}$. Then, since $3d = 9e$, we can choose any non-zero value for the free variable e, and d will be $3e$. So, if we choose 1 for e, then $d = 3e = 3$. Lastly, we must resolve $5a + 3b - 9c = 2$. Here, b and c are the free variables, and $a = \frac{2-3b+9c}{5}$. If we let $b = c = 1$. Then, $a = \frac{2-3(1)+9(1)}{5} = \frac{8}{5}$.

So, in our example, $A = \begin{bmatrix} \frac{8}{5} & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}$

1.3.**32** For this problem, we set up the same three equations as in Exercise 30. However, here, we must enforce that our matrix, A, contains no zero entries. One possible solution to this problem is the matrix $\begin{bmatrix} -2 & -2 & -2 \\ 3 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$.

1.3.33 The ith component of $A\vec{x}$ is $[0\ 0\ \dots\ 1\ \dots\ 0]\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_n \end{bmatrix} = x_i$. (The 1 is in the ith position.)

Therefore, $A\vec{x} = \vec{x}$.

1.3.34 a $A\vec{e}_1 = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$, $A\vec{e}_2 = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$, and $A\vec{e}_3 = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$.

b $B\vec{e}_1 = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{v}_1$.

Likewise, $B\vec{e}_2 = \vec{v}_2$ and $B\vec{e}_3 = \vec{v}_3$.

1.3.35 Write $A = [\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_i\ \dots\ \vec{v}_m]$, then

$$A\vec{e}_i = [\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_i\ \dots\ \vec{v}_m]\begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{bmatrix} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 1\vec{v}_i + \cdots + 0\vec{v}_m = \vec{v}_i = i\text{th column of } A.$$

1.3.36 By Exercise 35, the ith column of A is $A\vec{e}_i$, for $i = 1, 2, 3$. Therefore, $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

1.3.37 We have to solve the system $\begin{bmatrix} x_1 & + & 2x_2 & & = & 2 \\ & & & x_3 & = & 1 \end{bmatrix}$ or $\begin{bmatrix} x_1 & = 2 - 2x_2 \\ x_3 & = 1 \end{bmatrix}$.

Let $x_2 = t$. Then the solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 2t \\ t \\ 1 \end{bmatrix}$, where t is an arbitrary real number.

1.3.38 We will illustrate our reasoning with an example. We generate the "random" 3×3 matrix

$$A = \begin{bmatrix} 0.141 & 0.592 & 0.653 \\ 0.589 & 0.793 & 0.238 \\ 0.462 & 0.643 & 0.383 \end{bmatrix}.$$

Since the entries of this matrix are chosen from a large pool of numbers (in our case 1000, from 0.000 to 0.999), it is unlikely that any of the entries will be zero (and even less likely that the whole first column will consist of zeros). This means that we will usually be able to apply the Gauss-Jordan algorithm to turn the first column into $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; this is indeed possible in our example: $\begin{bmatrix} 0.141 & 0.592 & 0.653 \\ 0.589 & 0.793 & 0.238 \\ 0.462 & 0.643 & 0.383 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4.199 & 4.631 \\ 0 & -1.680 & -2.490 \\ 0 & -1.297 & -1.757 \end{bmatrix}$.

Again, it is unlikely that any entries in the second column of the new matrix will be zero. Therefore, we can turn the second column into $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Likewise, we will be able to clear up the third column, so that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We summarize:

As we apply Gauss-Jordan elimination to a random matrix A (of any size), it is unlikely that we will ever encounter a zero on the diagonal. Therefore, rref(A) is likely to have all ones along the diagonal.

1.3.**39** We will usually get $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$, where a, b, and c are arbitrary.

1.3.**40** We will usually have $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(Compare with the summary to Exercise 38.)

1.3.**41** If $A\vec{x} = \vec{b}$ is a "random" system, then rref(A) will usually be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that we will have a unique solution.

1.3.**42** If $A\vec{x} = \vec{b}$ is a "random" system of three equations with four unknowns, then rref(A) will usually be

$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ (by Exercise 39), so that the system will have infinitely many solutions (x_4 is a free variable).

1.3.**43** If $A\vec{x} = \vec{b}$ is a "random" system of equations with three unknowns, then $\text{rref}[A \vdots \vec{b}]$ will usually be

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$, so that the system is inconsistent.

1.3.**44** Let $E = \text{rref}(A)$, and note that all the entries in the last row of E must be zero, by the definition of rref. If $\vec{c}$ is any vector in $\mathbb{R}^n$ whose last component isn't zero, then the system $E\vec{x} = \vec{c}$ will be inconsistent. Now consider the elementary row operations that transform A into E, and apply the opposite operations, in reversed order, to the augmented matrix $\left[E \ \vdots \ \vec{c} \right]$. You end up with an augmented matrix $\left[A \ \vdots \ \vec{b} \right]$ that represents an inconsistent system $A\vec{x} = \vec{b}$, as required.

1.3.**45** Write $A = [\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_m]$ and $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix}$. Then $A(k\vec{x}) = [\vec{v}_1\ \dots\ \vec{v}_m] \begin{bmatrix} kx_1 \\ \dots \\ kx_m \end{bmatrix} = kx_1\vec{v}_1 + \cdots + kx_m\vec{v}_m$ and $k(A\vec{x}) = k(x_1\vec{v}_1 + \cdots + x_m\vec{v}_m) = kx_1\vec{v}_1 + \cdots + kx_m\vec{v}_m$. The two results agree, as claimed.

1.3.**46** Since a, d, and f are all nonzero, we can divide the first row by a, the second row by d, and the third row by f to obtain

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & 1 & \frac{e}{d} \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that the rank of the matrix is 3.

1.3.**47** a $\vec{x} = \vec{0}$ is a solution.

b This holds by part (a) and Theorem 1.3.3.

c If $\vec{x}_1$ and $\vec{x}_2$ are solutions, then $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$.

Therefore, $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$, so that $\vec{x}_1 + \vec{x}_2$ is a solution as well. Note that we have used Theorem 1.3.10a.

d $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$

We have used Theorem 1.3.10b.

1.3.**48** The fact that $\vec{x}_1$ is a solution of $A\vec{x} = \vec{b}$ means that $A\vec{x}_1 = \vec{b}$.

a. $A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = \vec{b} + \vec{0} = \vec{b}$

b. $A(\vec{x}_2 - \vec{x}_1) = A\vec{x}_2 - A\vec{x}_1 = \vec{b} - \vec{b} = \vec{0}$

c. Parts (a) and (b) show that the solutions of $A\vec{x} = \vec{b}$ are exactly the vectors of the form $\vec{x}_1 + \vec{x}_h$, where $\vec{x}_h$ is a solution of $A\vec{x} = \vec{0}$; indeed if $\vec{x}_2$ is a solution of $A\vec{x} = \vec{b}$, we can write $\vec{x}_2 = \vec{x}_1 + (\vec{x}_2 - \vec{x}_1)$, and $\vec{x}_2 - \vec{x}_1$ will be a solution of $A\vec{x} = \vec{0}$, by part (b).

Geometrically, the vectors of the form $\vec{x}_1 + \vec{x}_h$ are those whose tips are on the line L in Figure 1.14; the line L runs through the tip of $\vec{x}_1$ and is parallel to the given line consisting of the solutions of $A\vec{x} = \vec{0}$.

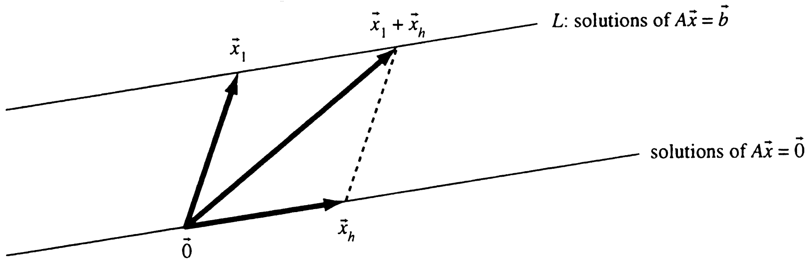


Figure 1.14: for Problem 1.3.48c.

1.3.**49** a This system has either infinitely many solutions (if the right-most column of rref$[A \vdots \vec{b}]$ does not contain a leading one), or no solutions (if the right-most column *does* contain a leading one).

b This system has either a unique solution (if rank$[A \vdots \vec{b}] = 3$), or no solution (if rank$[A \vdots \vec{b}] = 4$).

c The right-most column of rref$[A \vdots \vec{b}]$ must contain a leading one, so that the system has no solutions.

d This system has infinitely many solutions, since there is one free variable.

1.3.**50** The right-most column of rref$[A \vdots \vec{b}]$ must contain a leading one, so that the system has no solutions.

1.3.**51** For $B\vec{x}$ to be defined, the number of columns of B, which is s, must equal the number of components of $\vec{x}$, which is p, so that we must have $s = p$. Then $B\vec{x}$ will be a vector in $\mathbb{R}^r$; for $A(B\vec{x})$ to be defined we must have $m = r$. Summary: We must have $s = p$ and $m = r$.

1.3.**52** $A(B\vec{x}) = A\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} -x_2 \\ 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

so that $C = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$.

1.3.**53** Yes; write $A = [\vec{v}_1 \ \dots \ \vec{v}_m]$, $B = [\vec{w}_1 \ \dots \ \vec{w}_m]$, and $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix}$.

Then $(A+B)\vec{x} = [\vec{v}_1 + \vec{w}_1 \ \dots \ \vec{v}_m + \vec{w}_m]\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1(\vec{v}_1 + \vec{w}_1) + \cdots + x_m(\vec{v}_m + \vec{w}_m)$ and

$A\vec{x} + B\vec{x} = [\vec{v}_1 \ \dots \ \vec{v}_m]\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} + [\vec{w}_1 \ \dots \ \vec{w}_m]\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m + x_1\vec{w}_1 + \cdots + x_m\vec{w}_m$.

The two results agree, as claimed.

1.3.**54** The vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2$ form a plane through the origin containing $\vec{v}_1$ and $\vec{v}_2$; in Figure 1.15 we draw a typical vector in this plane.

1.3.**55** We are looking for constants a and b such that $a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

The resulting system $\begin{bmatrix} a + 4b & = 7 \\ 2a + 5b & = 8 \\ 3a + 6b & = 9 \end{bmatrix}$ has the unique solution $a = -1$, $b = 2$, so that $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ is indeed a linear combination of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

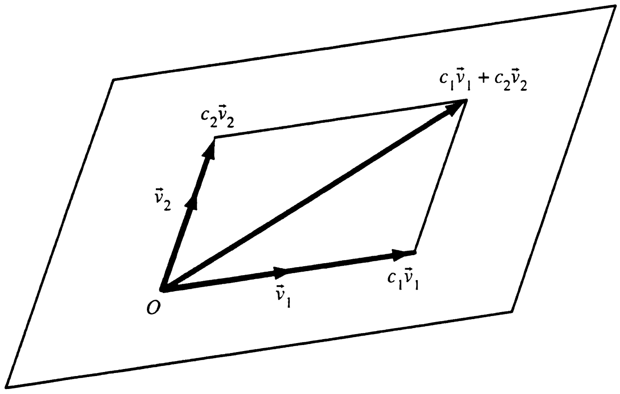


Figure 1.15: for Problem 1.3.54.

1.3.**56** We can use technology to determine that the system $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}$ is inconsistent; therefore, the vector $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$ fails to be a linear combination of the other four vectors.

1.3.**57** Pick a vector on each line, say $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ on $y = \frac{x}{2}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ on $y = 3x$.

Then write $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$: $a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$.

The unique solution is $a = 2,\ b = 3$, so that the desired representation is $\begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}$.

$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ is on the line $y = \frac{x}{2}$; $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ is on line $y = 3x$.

1.3.**58** We want $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + k_3 \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$, for some k_1, k_2 and k_3.

Note that we can rewrite this right-hand side as $k_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 2k_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - k_3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$= (k_1 + 2k_2 - k_3) \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. It follows that $k_1 + 2k_2 - k_3 = 3$, so that $b = 9$ and $c = 6$.

1.3.**59** $\begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} a+b \\ a+2b \\ a+3b \\ a+4b \end{bmatrix}$.

So we have a small system: $\begin{bmatrix} a & +b & = & 5 \\ a & +2b & = & 7 \end{bmatrix}$, which we quickly solve to find $a = 3$ and $b = 2$. Then, $c = a + 3b = 3 + 6 = 9$ and $d = a + 4b = 3 + 8 = 11$.

1.3.**60** We need $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} k_2 + 2k_3 \\ 0 \\ 3k_1 + 4k_2 + 5k_3 \\ 6k_3 \end{bmatrix}$. From this we see that a, c and d can be any value, while b must equal zero.

1.3.**61** We need to solve the system

$$\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

with augmented matrix

$$\begin{bmatrix} 1 & 1 & \vdots & 1 \\ 2 & 3 & \vdots & c \\ 4 & 9 & \vdots & c^2 \end{bmatrix}.$$

The matrix reduces to

$$\begin{bmatrix} 1 & 1 & \vdots & 1 \\ 0 & 1 & \vdots & c-2 \\ 0 & 0 & \vdots & c^2 - 5c + 6 \end{bmatrix}.$$

This system is consistent if and only if $c = 2$ or $c = 3$. Thus the vector is a linear combination if $c = 2$ or $c = 3$.

1.3.**62** We need to solve the system

$$\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix} = x \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} + y \begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$$

with augmented matrix

$$\begin{bmatrix} 1 & 1 & \vdots & 1 \\ a & b & \vdots & c \\ a^2 & b^2 & \vdots & c^2 \end{bmatrix}.$$

The matrix reduces to

$$\begin{bmatrix} 1 & 1 & \vdots & 1 \\ 0 & b-a & \vdots & c-a \\ 0 & 0 & \vdots & (c-a)(c-b) \end{bmatrix}.$$

This system is consistent if and only if $c = a$ or $c = b$. Thus the vector is a linear combination if $c = a$ or $c = b$.

1.3.**63** This is the line parallel to $\vec{w}$ which goes through the end point of the vector $\vec{v}$.

1.3.**64** This is the line segment connecting the head of the vector $\vec{v}$ to the head of the vector $\vec{v}+\vec{w}$.

1.3.**65** This is the full parallelogram spanned by the two vectors $\vec{v}$ and $\vec{w}$.

1.3.**66** Write $b = 1-a$ and $a\vec{v}+b\vec{w} = a\vec{v}+(1-a)\vec{w} = \vec{w}+a(\vec{v}-\vec{w})$ to see that this is the line segment connecting the head of the vector $\vec{v}$ to the head of the vector $\vec{w}$.

1.3.**67** This is the full triangle with its vertices at the origin and at the heads of the vectors $\vec{v}$ and $\vec{w}$.

1.3.**68** Writing $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w}$ as $\vec{u}\cdot(\vec{v}-\vec{w}) = 0$, we see that this is the line perpendicular to the vector $\vec{v}-\vec{w}$.

1.3.**69** We write out the augmented matrix: $\begin{bmatrix} 0 & 1 & 1 & \vdots & a \\ 1 & 0 & 1 & \vdots & b \\ 1 & 1 & 0 & \vdots & c \end{bmatrix}$ and reduce it to $\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & \vdots & \frac{a-b+c}{2} \\ 0 & 0 & 1 & \vdots & \frac{a+b-c}{2} \end{bmatrix}$.

So $x = \frac{-a+b+c}{2}$, $y = \frac{a-b+c}{2}$ and $z = \frac{a+b-c}{2}$.

1.3.**70** We find it useful to let $s = x_1 + x_2 + \cdots + x_n$. Adding up all n equations of the system, and realizing that the term x_i is missing from the i^{th} equation, we see that $(n-1)s = b_1 + \cdots + b_n$, or, $s = \frac{b_1+\cdots+b_n}{n-1}$. Now the i^{th} equation of the system can be written as $s - x_i = b_i$, so that $x_i = s - b_i = \frac{b_1+\cdots+b_n}{n-1} - b_i$.

True or False

Ch 1.TF.**1** T, by Theorem 1.3.8

Ch 1.TF.**2** T, by Definition 1.3.9

Ch 1.TF.**3** T, by Definition.

Ch 1.TF.**4** F; Consider the equation $x + y + z = 0$, repeated four times.

Ch 1.TF.**5** F, by Example 3a of Section 1.3

Ch 1.TF.**6** T, by Definition 1.3.7

Ch 1.TF.**7** T, by Theorem 1.3.4

Ch 1.TF.**8** F, by Theorem 1.3.1

Ch 1.TF.**9** F, by Theorem 1.3.4

Ch 1.TF.**10** F; As a counter-example, consider the zero matrix.

Ch 1.TF.**11** T; The last component of the left-hand side is zero for all vectors $\vec{x}$.

Ch 1.TF.**12** T; $A = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$, for example.

Ch 1.TF.**13** T; Find rref

Ch 1.TF.**14** T; Find rref

Ch 1.TF.**15** F; Consider the 4×3 matrix A that contains all zeroes, except for a 1 in the lower left corner.

Ch 1.TF.**16** F; Note that $A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for all 2×2 matrices A.

Ch 1.TF.**17** F; The rank is 1.

Ch 1.TF.**18** F; The product on the left-hand side has two components.

Ch 1.TF.**19** T; Let $A = \begin{bmatrix} -3 & 0 \\ -5 & 0 \\ -7 & 0 \end{bmatrix}$, for example.

Ch 1.TF.**20** T; We have $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

Ch 1.TF.**21** F; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, for example.

Ch 1.TF.**22** T, by Exercise 1.3.44.

Ch 1.TF.**23** F; Find rref to see that the rank is always 2.

Ch 1.TF.**24** T; Note that $\vec{v} = 1\vec{v} + 0\vec{w}$.

Ch 1.TF.**25** F; Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.

Ch 1.TF.**26** T; Note that $\vec{0} = 0\vec{v} + 0\vec{w}$

Ch 1.TF.**27** F; Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for example. We can apply elementary row operations to A all we want, we will always end up with a matrix that has all zeros in the first column.

Ch 1.TF.**28** T; If $\vec{u} = a\vec{v} + b\vec{w}$ and $\vec{v} = c\vec{p} + d\vec{q} + e\vec{r}$, then $\vec{u} = ac\vec{p} + ad\vec{q} + ae\vec{r} + b\vec{w}$.

Ch 1.TF.**29** F; The system $x = 2,\ y = 3,\ x + y = 5$ has a unique solution.

Ch 1.TF.**30** F; Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example.

Ch 1.TF.**31** F; If $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{0}$, then $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a solution to $\left[A \vdots \vec{0}\right]$. However, since $\text{rank}(A) = 3$, rref $\left[A \vdots \vec{0}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, meaning that only $\vec{0}$ is a solution to $A\vec{x} = \vec{0}$.

Ch 1.TF.**32** F; If $\vec{b} = \vec{0}$, then having a row of zeroes in rref(A) does not force the system to be inconsistent.

Ch 1.TF.**33** T; By Example 4d of Section 1.3, the equation $A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$. Now note that $A(\vec{v} - \vec{w}) = \vec{0}$, so that $\vec{v} - \vec{w} = \vec{0}$ and $\vec{v} = \vec{w}$.

Ch 1.TF.**34** T; Note that $\text{rank}(A) = 4$, by Theorem 1.3.4

Ch 1.TF.**35** F; Let $\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example..

Ch 1.TF.**36** T; We use rref to solve the system $A\vec{x} = \vec{0}$ and find $\vec{x} = \begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix}$, where t is an arbitrary constant. Letting $t = 1$, we find $[\vec{u}\ \vec{v}\ \vec{w}] \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = -2\vec{u} - 3\vec{v} + \vec{w} = \vec{0}$, so that $\vec{w} = 2\vec{u} + 3\vec{v}$.

Ch 1.TF.**37** F; Let $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, for example.

Ch 1.TF.**38** T; Matrices A and B can both be transformed into $I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Running the elementary operations backwards, we can transform I into B. Thus we can first transform A into I and then I into B.

Ch 1.TF.**39** T; If $\vec{v} = a\vec{u} + b\vec{w}$, then $A\vec{v} = A(a\vec{u} + b\vec{w}) = A(a\vec{u}) + A(b\vec{w}) = aA\vec{v} + bA\vec{w}$.

Ch 1.TF.**40** T; check that the three defining properties of a matrix in rref still hold. F; If $\vec{b} = \vec{0}$, then having a row of zeroes in rref(A) does not force the system to be inconsistent.

Ch 1.TF.**41** T; $A\vec{x} = \vec{b}$ is inconsistent if and only if rank$\left[A \vdots \vec{b}\right] = \text{rank}(A)+1$, since there will be an extra leading one in the last column of the augmented matrix: (See Figure 1.16.)

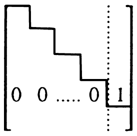


Figure 1.16: for Problem T/F 41.

Ch 1.TF.**42** T; The system $A\vec{x} = \vec{b}$ is consistent, by Example 4b, and there are, in fact, infinitely many solutions, by Example 4c. Note that $A\vec{x} = \vec{b}$ is a system of three equations with four unknowns.

Ch 1.TF.**43** T; Recall that we use $\text{rref}\left[A \vdots \vec{0}\right]$ to solve the system $A\vec{x} = \vec{0}$. Now, $\text{rref}\left[A \vdots \vec{0}\right] = \left[\text{rref}(A) \vdots \vec{0}\right] = \left[\text{rref}(B) \vdots \vec{0}\right] = \text{rref}\left[B \vdots \vec{0}\right]$. Then, since $\left[\text{rref}(A) \vdots \vec{0}\right] = \left[\text{rref}(B) \vdots \vec{0}\right]$, they must have the same solutions.

Ch 1.TF.**44** F; Consider $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. If we remove the first column, then the remaining matrix fails to be in rref.

Ch 1.TF.**45** T; First we list all possible matrices rref(M), where M is a 2×2 matrix, and show the corresponding solutions for $M\vec{x} = \vec{0}$:

rref(M)	solutions of $M\vec{x} = \vec{0}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\{\vec{0}\}$
$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -at \\ t \end{bmatrix}$, for an arbitrary t
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} t \\ 0 \end{bmatrix}$, for an arbitrary t
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\mathbb{R}^2$

Now, we see that if $\text{rref}(A) \neq \text{rref}(B)$, then the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ must have different solutions. Thus, it must be that if the two systems have the same solutions, then $\text{rref}(A) = \text{rref}(B)$.

Ch 1.TF.**46** T . First note that the product of the diagonal entries is nonzero if (and only if) all three diagonal entries are nonzero.

If all the diagonal entries are nonzero, then $A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{matrix} \div a \\ \div c \\ \div f \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ b' & 1 & 0 \\ d' & e' & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, showing that rank $A = 3$.

Conversely, if $a = 0$ or $c = 0$ or $f = 0$, then it is easy to verify that rref A will contain a row of zeros, so that rank $A \leq 2$. For example, if a and c are nonzero but $f = 0$, then

$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & 0 \end{bmatrix} \begin{matrix} \div a \\ \div c \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ b' & 1 & 0 \\ d & e & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, with rank $A = 2$.

Ch 1.TF.**47** T. If $a \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} \div a \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \begin{matrix} \\ -c(I) \end{matrix} \rightarrow \begin{bmatrix} 1 & b/a \\ 0 & (ad-bc)/a \end{bmatrix} \begin{matrix} \\ a/(ad-bc) \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, showing that rank $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2$.

If $a = 0$, then b and c are both nonzero, so that $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as claimed.

Ch 1.TF.**48** T. If $\vec{w} = a\vec{u} + b\vec{v}$, then $\vec{u} + \vec{v} + \vec{w} = (a+1)\vec{u} + (b+1)\vec{v} = (a-b)\vec{u} + (b+1)(\vec{u} + \vec{v})$

Ch 1.TF.**49** T. If $A\vec{v} = \vec{b}$ and $A\vec{w} = \vec{c}$, then $A(\vec{v} + \vec{w}) = \vec{b} + \vec{c}$, showing that the system $A\vec{x} = \vec{b} + \vec{c}$ is consistent. Suppose A is an $n \times m$ matrix. Since $A\vec{x} = \vec{b}$ has a unique solution, rank A must be m (by Example 1.3.3c), implying that the system $A\vec{x} = \vec{b} + \vec{c}$ has a unique solution as well (by Example 1.3.4d).

Ch 1.TF.**50** F. Think about constructing a 0-1 matrix A of size 3×3 with rank $A = 3$ row by row. The rows must be chosen so that rref A will not contain a row of zeros, which implies that no two rows of A can be equal. For the first row we have $7 = 2^3 - 1$ choices: anything except $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. For the second row we have six choices left: anything except the row of zeros and the first row. For the third row we have at most five choices, since we cannot chose the row of zeros, the first row, or the second row. Thus, at most $7 \times 6 \times 5 = 210$ of the 0-1-matrices of size 3×3 have rank 3, out of a total of $2^9 = 512$ matrices.