Instructor's Manual to Accompany

Chapter 1

1. $S = \{(R, R), (R, G), (R, B), (G, R), (G, G), (G, B), (B, R), (B, G), (B, B)\}$
   The probability of each point in S is 1/9.
2. $S = \{(R, G), (R, B), (G, R), (G, B), (B, R), (B, G)\}$
3. $S = \{(e_1, e_2, \ldots, e_n),\ n \geq 2\}$ where $e_i \in$(heads, tails}. In addition, $e_n = e_{n-1} =$ heads and for $i = 1, \ldots, n-2$ if $e_i =$ heads, then $e_{i+1} =$ tails.

$$\begin{aligned} P\{4 \text{ tosses}\} &= P\{(t, t, h, h)\} + P\{(h, t, h, h)\} \\ &= 2\left[\frac{1}{2}\right]^4 = \frac{1}{8} \end{aligned}$$

4. (a) $F(E \cup G)^c = FE^cG^c$
   (b) EFG^c
   (c) $E \cup F \cup G$
   (d) $EF \cup EG \cup FG$
   (e) EFG
   (f) $(E \cup F \cup G)^c = E^cF^cG^c$
   (g) $(EF)^c(EG)^c(FG)^c$
   (h) $(EFG)^c$
5. $\frac{3}{4}$. If he wins, he only wins \$1, while if he loses, he loses \$3.
6. If $E(F \cup G)$ occurs, then E occurs and either F or G occur; therefore, either EF or EG occurs and so

$$E(F \cup G) \subset EF \cup EG$$

Similarly, if $EF \cup EG$ occurs, then either EF or EG occurs. Thus, E occurs and either F or G occurs; and so $E(F \cup G)$ occurs. Hence,

$$EF \cup EG \subset E(F \cup G)$$

which together with the reverse inequality proves the result.

7. If $(E \cup F)^c$ occurs, then $E \cup F$ does not occur, and so E does not occur (and so E^c does); F does not occur (and so F^c does) and thus E^c and F^c both occur. Hence,

$$(E \cup F)^c \subset E^c F^c$$

If $E^c F^c$ occurs, then E^c occurs (and so E does not), and F^c occurs (and so F does not). Hence, neither E or F occurs and thus $(E \cup F)^c$ does. Thus,

$$E^c F^c \subset (E \cup F)^c$$

and the result follows.

8. $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$
9. $F = E \cup FE^c$, implying since E and FE^c are disjoint that $P(F) = P(E) + P(FE)^c$.
10. Either by induction or use

$$\bigcup_1^n E_i = E_1 \cup E_1^c E_2 \cup E_1^c E_2^c E_3 \cup \cdots \cup E_1^c \cdots E_{n-1}^c E_n$$

and as each of the terms on the right side are mutually exclusive:

$$\begin{aligned} P(\bigcup_{\mathrm{i}} E_i) &= P(E_1) + P(E_1^c E_2) + P(E_1^c E_2^c E_3) + \cdots \\ &\quad + P(E_1^c \cdots E_{n-1}^c E_n) \\ &\leq P(E_1) + P(E_2) + \cdots + P(E_n) \quad \text{(why?)} \end{aligned}$$

11. $P\{\text{sum is } i\} = \begin{cases} \frac{i-1}{36}, & i = 2, \ldots, 7 \\ \frac{13-i}{36}, & i = 8, \ldots, 12 \end{cases}$
12. Either use hint or condition on initial outcome as:

$$\begin{aligned} &P\{E \text{ before } F\} \\ &\quad = P\{E \text{ before } F | \text{initial outcome is } E\} P(E) \\ &\qquad + P\{E \text{ before } F | \text{initial outcome is } F\} P(F) \\ &\qquad + P\{E \text{ before } F | \text{initial outcome neither E or } F\}[1 - P(E) - P(F)] \\ &\quad = 1 \cdot P(E) + 0 \cdot P(F) + P\{E \text{ before } F\} \\ &\quad = [1 - P(E) - P(F)] \end{aligned}$$

Therefore, $P\{E \text{ before } F\} = \frac{P(E)}{P(E)+P(F)}$

13. Condition an initial toss

$$P\{\text{win}\} = \sum_{i=2}^{12} P\{\text{win}|\text{throw } i\} P\{\text{throw } i\}$$

Now,

$$P\{\text{win}|\text{throw } i\} = P\{i \text{ before } 7\}$$
$$= \begin{cases} 0 & i = 2, 12 \\ \dfrac{i-1}{5+1} & i = 3, \ldots, 6 \\ 1 & i = 7, 11 \\ \dfrac{13-i}{19-1} & i = 8, \ldots, 10 \end{cases}$$

where above is obtained by using Problems 11 and 12.

$$P\{\text{win}\} \approx .49.$$

14. $$P\{A \text{ wins}\} = \sum_{n=0}^{\infty} P\{A \text{ wins on } (2n+1)\text{st toss}\}$$
$$= \sum_{n=0}^{\infty} (1-P)^{2n} P$$
$$= P \sum_{n=0}^{\infty} [(1-P)^2]^n$$
$$= P \frac{1}{1-(1-P)^2}$$
$$= \frac{P}{2P - P^2}$$
$$= \frac{1}{2-P}$$
$$P\{B \text{ wins}\} = 1 - P\{A \text{ wins}\}$$
$$= \frac{1-P}{2-P}$$

16. $$P(E \cup F) = P(E \cup FE^c)$$
$$= P(E) + P(FE^c)$$

since E and FE^c are disjoint. Also,

$$P(E) = P(FE \cup FE^c)$$
$$= P(FE) + P(FE^c) \text{ by disjointness}$$

Hence,

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

17. $$\text{Prob}\{\text{end}\} = 1 - \text{Prob}\{\text{continue}\}$$
$$= 1 - P(\{H, H, H\} \cup \{T, T, T\})$$
$$= 1 - [\text{Prob}(H, H, H) + \text{Prob}(T, T, T)].$$

$$\text{Fair coin: Prob}\{\text{end}\} = 1 - \left[\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right]$$
$$= \frac{3}{4}$$
$$\text{Biased coin:} P\{\text{end}\} = 1 - \left[\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right]$$
$$= \frac{9}{16}$$

18. Let B = event both are girls; E = event oldest is girl; L = event at least one is a girl.

(a) $P(B|E) = \dfrac{P(BE)}{P(E)} = \dfrac{P(B)}{P(E)} = \dfrac{1/4}{1/2} = \dfrac{1}{2}$

(b) $P(L) = 1 - P(\text{no girls}) = 1 - \dfrac{1}{4} = \dfrac{3}{4}$,

$P(B|L) = \dfrac{P(BL)}{P(L)} = \dfrac{P(B)}{P(L)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$

19. E = event at least 1 six $P(E)$

$= \dfrac{\text{number of ways to get } E}{\text{number of samples pts}} = \dfrac{11}{36}$

D = event two faces are different $P(D)$

$= 1 -$ Prob(two faces the same)

$= 1 - \dfrac{6}{36} = \dfrac{5}{6} P(E|D) = \dfrac{P(ED)}{P(D)} = \dfrac{10/36}{5/6} = \dfrac{1}{3}$

20. Let E = event same number on exactly two of the dice; S = event all three numbers are the same; D = event all three numbers are different. These three events are mutually exclusive and define the whole sample space. Thus, $1 = P(D) + P(S) + P(E)$, $P(S) = 6/216 = 1/36$; for D have six possible values for first die, five for second, and four for third.

$\therefore$ Number of ways to get $D = 6 \cdot 5 \cdot 4 = 120$.

$P(D) = 120/216 = 20/36$

$\therefore P(E) = 1 - P(D) - P(S)$

$= 1 - \dfrac{20}{36} - \dfrac{1}{36} = \dfrac{5}{12}$

21. Let C = event person is color blind.

$$P(\text{Male}|C) = \frac{P(C|\text{Male})P(\text{Male})}{P(C|\text{Male}P(\text{Male}) + P(C|\text{Female})P(\text{Female})}$$
$$= \frac{.05 \times .5}{.05 \times .5 + .0025 \times .5}$$
$$= \frac{2500}{2625} = \frac{20}{21}$$

22. Let trial 1 consist of the first two points; trial 2 the next two points, and so on. The probability that each player wins one point in a trial is $2p(1-p)$. Now a total of $2n$ points are played if the first $(a-1)$ trials all result in each player winning one of the points in that trial and the nth trial results in one of the players winning both points. By independence, we obtain

$$\begin{aligned}&P\{2n \text{ points are needed}\}\\&= (2p(1-p))^{n-1}(p^2+(1-p)^2), \quad n \geq 1\end{aligned}$$

The probability that A wins on trial n is $(2p(1-p))^{n-1}p^2$ and so

$$\begin{aligned}P\{A \text{ wins}\} &= p^2 \sum_{n=1}^{\infty}(2p(1-p))^{n-1}\\&= \frac{p^2}{1-2p(1-p)}\end{aligned}$$

23. $P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1\ldots E_{n-1})$

$$\begin{aligned}&= P(E_1)\frac{P(E_1E_2)}{P(E_1)}\frac{P(E_1E_2E_3)}{P(E_1E_2)}\cdots\frac{P(E_1\ldots E_n)}{P(E_1\ldots E_{n-1})}\\&= P(E_1\ldots E_n)\end{aligned}$$

24. Let a signify a vote for A and b one for B.

(a) $P_{2,1} = P\{a,a,b\} = 1/3$
(b) $P_{3,1} = P\{a,a\} = (3/4)(2/3) = 1/2$
(c) $P_{3,2} = P\{a,a,a\} + P\{a,a,b,a\}$
$= (3/5)(2/4)[1/3 + (2/3)(1/2)] = 1/5$

(d) $P_{4,1} = P\{a,a\} = (4/5)(3/4) = 3/5$
(e) $P_{4,2} = P\{a,a,a\} + P\{a,a,b,a\}$
$= (4/6)(3/5)[2/4 + (2/4)(2/3)] = 1/3$

(f)
$$\begin{aligned}P_{4,3} &= P\{\text{always ahead}|a,a\}(4/7)(3/6)\\&= (2/7)[1 - P\{a,a,a,b,b,b|a,a\}\\&\quad - P\{a,a,b,b|a,a\} - P\{a,a,b,a,b,b|a,a\}]\\&= (2/7)[1-(2/5)(3/4)(2/3)(1/2)\\&\quad -(3/5)(2/4)-(3/5)(2/4)(2/3)(1/2)]\\&= 1/7\end{aligned}$$

(g) $P_{5,1} = P\{a,a\} = (5/6)(4/5) = 2/3$
(h) $P_{5,2} = P\{a,a,a\} + P\{a,a,b,a\}$
$= (5/7)(4/6)[(3/5) + (2/5)(3/4)] = 3/7$

By the same reasoning we have

(i) $P_{5,3} = 1/4$
(j) $P_{5,4} = 1/9$
(k) In all the cases above, $P_{n,m} = \frac{n-n}{n+n}$

25. (a) $P\{pair\} = P\{\text{second card is same denomination as first}\}$
$$= 3/51$$
(b) $P\{\text{pair}|\text{different suits}\}$
$$= \frac{P\{\text{pair, different suits}\}}{P\{\text{different suits}\}}$$
$$= P\{\text{pair}\}/P\{\text{different suits}\}$$
$$= \frac{3/51}{39/51} = 1/13$$

26. $$P(E_1) = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} = \frac{39.38.37}{51.50.49}$$
$$P(E_2|E_1) = \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} = \frac{26.25}{38.37}$$
$$P(E_3|E_1E_2) = \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13} = 13/25$$
$$P(E_4|E_1E_2E_3) = 1$$
$$P(E_1E_2E_3E_4) = \frac{39.26.13}{51.50.49}$$

27. $P(E_1) = 1$
$P(E_2|E_1) = 39/51$, since 12 cards are in the ace of spades pile and 39 are not.
$P(E_3|E_1E_2) = 26/50$, since 24 cards are in the piles of the two aces and 26 are in the other two piles.
$$P(E_4|E_1E_2E_3) = 13/49$$
So
$$P\{\text{each pile has an ace}\} = (39/51)(26/50)(13/49)$$

28. Yes. $P(A|B) > P(A)$ is equivalent to $P(AB) > P(A)P(B)$, which is equivalent to $P(B|A) > P(B)$.

29. (a) $P(E|F) = 0$
(b) $P(E|F) = P(EF)/P(F) = P(E)/P(F) \geq P(E) = .6$
(c) $P(E|F) = P(EF)/P(F) = P(F)/P(F) = 1$

30. (a) $$P\{\text{George}|\text{exactly 1 hit}\} = \frac{P\{\text{George, not Bill}\}}{P\{\text{exactly 1}\}}$$
$$= \frac{P\{G,\ \text{not } B\}}{P\{G,\ \text{not } B\} + P\{B,\ \text{not } G)\}}$$
$$= \frac{(.4)(.3)}{(.4)(.3) + (.7)(.6)}$$
$$= 2/9$$

(b)
$$\begin{aligned} P\{G|\text{hit}\} &= P\{G,\ \text{hit}\}/P\{\text{hit}\} \\ &= P\{G\}/P\{\text{hit}\} = .4/[1-(.3)(.6)] \\ &= 20/41 \end{aligned}$$

31. Let S = event sum of dice is 7; F = event first die is 6.

$$\begin{aligned} P(S) &= \frac{1}{6} P(FS) = \frac{1}{36} P(F|S) = \frac{P(F|S)}{P(S)} \\ &= \frac{1/36}{1/6} = \frac{1}{6} \end{aligned}$$

32. Let E_i = event person i selects own hat. P (no one selects own hat)

$$\begin{aligned} &= 1 - P(E_1 \cup E_2 \cup \cdots \cup E_n) \\ &= 1 - \Big[\sum_{i_1} P(Ei_1) - \sum_{i_1<i_2} P(Ei_1 Ei_2) + \cdots \\ &\quad + (-1)^{n+1} P(E_1 E_2 E_n)\Big] \\ &= 1 - \sum_{i_1} P(Ei_1) - \sum_{i_1<i_2} P(Ei_1 Ei_2) \\ &\quad - \sum_{i_1<i_2<i_3} P(Ei_1 Ei_2 Ei_3) + \cdots \\ &\quad + (-1)^n P(E_1 E_2 E_n) \end{aligned}$$

Let $k \in \{1, 2, \ldots, n\}$. $P(Ei_1 EI_2 Ei_k)$ = number of ways k specific men can select own hats ÷ total number of ways hats can be arranged $= (n-k)!/n!$. Number of terms in summation $\sum_{i_1<i_2<\cdots<i_k}$ = number of ways to choose k variables out of n variables $= \begin{bmatrix} n \\ k \end{bmatrix} = n!/k!(n-k)!$.

Thus,

$$\begin{aligned} &\sum_{i_1<\cdots<i_k} P(Ei_1 Ei_2 \cdots Ei_k) \\ &\quad = \sum_{i_1<\cdots<i_k} \frac{(n-k)!}{n!} \\ &\quad = \begin{bmatrix} n \\ k \end{bmatrix} \frac{(n-k)!}{n!} = \frac{1}{k!} \end{aligned}$$

$\therefore$ P(no one selects own hat)

$$\begin{aligned} &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \\ &= \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \end{aligned}$$

33. Note that each set played is independently golden with probability $2(1/2)^{24} = (1/2)^{23}$. Condition on whether the match has 2 or 3 sets. Using that there will be a total of 2 sets with probability 1/2 yields the solution: $(1-(1-(1/2)^{23})^2)(1/2) + (1-(1-(1/2)^{23})^3)(1/2)$.
It should be noted that the preceding uses that whether a set is golden is independent of who won the set. That is, the conditional probability that a set is golden given that it is won by player 1 is $(1/2)^{23}$. To see why, it is obvious that who wins is independent of the event that the set was golden (since the winner given that it was golden is still equally likely to be either 1 or 2). But independence is symmetric, thus the event that a set is golden is independent of who won the set.

34. Letting A and B be respectively the events that the computer is fixed by A and that it is fixed by B. As A and B are mutually exclusive

$$P(A \cup B) = .4 + .6(.2) = .52$$

35. (a) 1/16
(b) 1/16
(c) 15/16, since the only way in which the pattern H, H, H, H can appear before the pattern T, H, H, H is if the first four flips all land heads.

36. Let B = event marble is black; B_i = event that box i is chosen. Now

$$\begin{aligned} B &= BB_1 \cup BB_2 P(B) = P(BB_1) + P(BB_2) \\ &= P(B|B_1)P(B_1) + P(B|B_2)P(B_2) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} = \frac{7}{12} \end{aligned}$$

37. Let W = event marble is white.

$$\begin{aligned} P(B_1|W) &= \frac{P(W|B_1)P(B_1)}{P(W|B_1)P(B_1) + P(W|B_2)P(B_2)} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5} \end{aligned}$$

38. Let T_W = event transfer is white; T_B = event transfer is black; W = event white ball is drawn from urn 2.

$$\begin{aligned} P(T_W|W) &= \frac{P(W|T_W)P(T_W)}{P(W|T_W)P(T_W) + P(W|T_B)P(T_B)} \\ &= \frac{\frac{2}{7}\cdot\frac{2}{3}}{\frac{2}{7}\cdot\frac{2}{3} + \frac{1}{7}\cdot\frac{1}{3}} = \frac{\frac{4}{21}}{\frac{5}{21}} = \frac{4}{5} \end{aligned}$$

39. Let W = event woman resigns; A, B, C are events the person resigning works in store A, B, C, respectively.

$$P(C|W) = \frac{P(W|C)P(C)}{P(W|C)P(C) + P(W|B)P(B) + P(W|A)P(A)}$$

$$= \frac{.70 \times \frac{100}{225}}{.70 \times \frac{100}{225} + .60 \times \frac{75}{225} + .50 \times \frac{50}{225}}$$

$$= \frac{70}{225} \bigg/ \frac{140}{225} = \frac{1}{2}$$

40. (a) F = event fair coin flipped; U = event two-headed coin flipped.

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(b) $$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH|F)P(F) + P(HH|U)P(U)}$$

$$= \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{\frac{1}{8}}{\frac{5}{8}} = \frac{1}{5}$$

(c) $$P(F|HHT) = \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + P(HHT|U)P(U)}$$

$$= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + 0} = 1$$

since the fair coin is the only one that can show tails.

41. Note first that since the rat has black parents and a brown sibling, we know that both its parents are hybrids with one black and one brown gene (for if either were a pure black then all their offspring would be black). Hence, both of their offspring's genes are equally likely to be either black or brown.

(a) $$P(\text{2 black genes}|\text{at least one black gene}) = \frac{P(\text{2 black genes})}{P(\text{at least one black gene})}$$

$$= \frac{1/4}{3/4} = 1/3$$

(b) Using the result from part (a) yields the following:

$$P(\text{2 black genes}|\text{5 black offspring}) = \frac{P(\text{2 black genes})}{P(\text{5 black offspring})}$$

$$= \frac{1/3}{1(1/3) + (1/2)^5(2/3)}$$

$$= 16/17$$

where P(5 black offspring) was computed by conditioning on whether the rat had 2 black genes.

42. Let B = event biased coin was flipped; F and U (same as above).

$$P(U|H) = \frac{P(H|U)P(U)}{P(H|U)P(U) + P(H|B)P(B) + P(H|F)P(F)}$$
$$= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{9}{12}} = \frac{4}{9}$$

43. Let B be the event that Flo has a blue eyed gene. Using that Jo and Joe both have one blue-eyed gene yields, upon letting X be the number of blue-eyed genes possessed by a daughter of theirs, that

$$P(B) = P(X = 1|X < 2) = \frac{1/2}{3/4} = 2/3$$

Hence, with C being the event that Flo's daughter is blue eyed, we obtain

$$P(C) = P(CB) = P(B)P(C|B) = 1/3$$

44. Let W = event white ball selected.

$$P(T|W) = \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|H)P(H)}$$
$$= \frac{\frac{1}{5} \cdot \frac{1}{2}}{\frac{1}{5} \cdot \frac{1}{2} + \frac{5}{12} \cdot \frac{1}{2}} = \frac{12}{37}$$

45. Let B_i = event ith ball is black; R_i = event ith ball is red.

$$P(B_1|R_2) = \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)}$$
$$= \frac{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}{\frac{r}{b+r+c} \cdot \frac{b}{b+r} + \frac{r+c}{b+r+c} \cdot \frac{r}{b+r}}$$
$$= \frac{rb}{rb + (r+c)r}$$
$$= \frac{b}{b+r+c}$$

46. Let $X (= B$ or $= C)$ denote the jailer's answer to prisoner A. Now for instance,

$$P\{A \text{ to be executed}|X = B\}$$
$$= \frac{P\{A \text{ to be executed}, X = B\}}{P\{X = B\}}$$
$$= \frac{P\{A \text{ to be executed}\}P\{X = B|A \text{ to be executed}\}}{P\{X = B\}}$$
$$= \frac{(1/3)P\{X = B|A \text{ to be executed}\}}{1/2}.$$

Now it is reasonable to suppose that if A is to be executed, then the jailer is equally likely to answer either B or C. That is,

$$P\{X = B|A \text{ to be executed}\} = \frac{1}{2}$$

and so,

$$P\{A \text{ to be executed}|X = B\} = \frac{1}{3}$$

Similarly,

$$P\{A \text{ to be executed}|X = C\} = \frac{1}{3}$$

and thus the jailer's reasoning is invalid. (It is true that if the jailer were to answer B, then A knows that the condemned is either himself or C, but it is twice as likely to be C.)

47. 1. $0 \le P(A|B) \le 1$
 2. $P(S|B) = \dfrac{P(SB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$
 3. For disjoint events A and D

$$\begin{aligned} P(A \cup D|B) &= \frac{P((A \cup D)B)}{P(B)} \\ &= \frac{P(AB \cup DB)}{P(B)} \\ &= \frac{P(AB) + P(DB)}{P(B)} \\ &= P(A|B) + P(D|B) \end{aligned}$$

 Direct verification is as follows:

$$\begin{aligned} &P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B) \\ &\quad = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(B)} + \frac{P(ABC^c)}{P(BC^c)}\frac{P(BC^c)}{P(B)} \\ &\quad = \frac{P(ABC)}{P(B)} + \frac{P(ABC^c)}{P(B)} \\ &\quad = \frac{P(AB)}{P(B)} \\ &\quad = P(A|B) \end{aligned}$$

49. Apply Proposition 1.1 to the increasing events $A_n^c, n \ge 1$.
50. (a) It is always the case that

$$\liminf A_n \subset \limsup A_n \subset \cup_n A_n$$