CHAPTER TWO

Instructor's Solutions Manual

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FOR

A Course In

Probability

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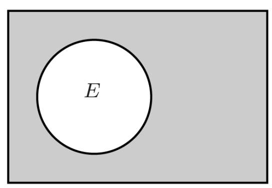
Chapter 2

Mathematical Probability

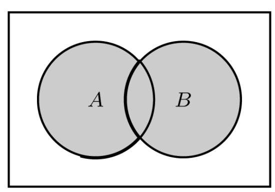
2.1 Sample Space and Events

Basic Exercises

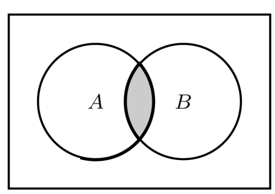
2.1 In each case, the shaded region represents the event in question.
a) E^c is the event that E does not occur:



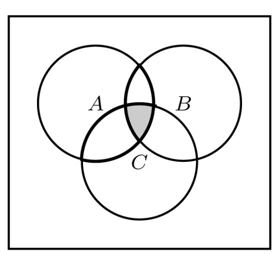
b) $A \cup B$ is the event that either A or B or both occur (i.e., at least one of A and B occurs):



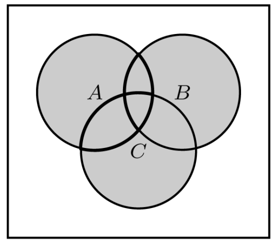
c) $A \cap B$ is the event that both A and B occur:



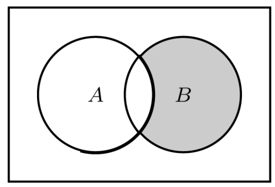
d) $A \cap B \cap C$ is the event that all three of A, B, and C occur:



e) $A \cup B \cup C$ is the event that at least one of A, B, and C occurs:



f) $A^c \cap B$ is the event that B occurs but A doesn't occur:



2.2
a) The statement is false. For instance, suppose that $A \neq \emptyset$ and $A \cap B = \emptyset$. Let C be any event such that $A \cap C \neq \emptyset$ (e.g., $C = A$). Then, although, $A \cap B = \emptyset$, events A, B, and C are not mutually exclusive because $A \cap C \neq \emptyset$.
b) The statement is true. Indeed, if A and B are not mutually exclusive, then $A \cap B \neq \emptyset$. Therefore, for any event C, events A, B, and C are not mutually exclusive because $A \cap B \neq \emptyset$.

2.3
a) The distance of the (center of the first) spot from the center of the Petri dish can be any real number between 0 (inclusive) and 1 (exclusive). Thus, a sample space for this random experiment is $[0, 1)$.
b) The event that the spot is between 1/4 and 1/2 unit, inclusive, from the center of the petri dish is $[1/4, 1/2]$.
c) The event $[0, 1/3]$ is that the spot is at most $1/3$ unit from the center of the petri dish.

2.4
a) A sample space for this random experiment is $\{1, 2, 3, 4, 5, 6\}$, where, for instance, "4" represents the outcome that four dots are facing up.
b) The die comes up even if and only if the outcome is 2, 4, or 6; thus, $A = \{2, 4, 6\}$. The die comes up at least 4 if and only if the outcome is 4, 5, or 6; thus, $B = \{4, 5, 6\}$. The die comes up at most 2 if and only if the outcome is either 1 or 2; thus, $C = \{1, 2\}$. The die comes up 3 if and only if the outcome is 3; thus, $D = \{3\}$.
c) We have $A^c = \{1, 3, 5\}$, which is the event that the die comes up odd; $A \cap B = \{4, 6\}$, which is the event that the die comes up either 4 or 6; $B \cup C = \{1, 2, 4, 5, 6\}$, which is the event that the die does not come up 3.
d) Events A and B are not mutually exclusive because they both occur when the die comes up either 4 or 6. Events B and C are mutually exclusive because it's impossible for the die to come up at least 4 and at most 2. Events A, C, and D are not mutually exclusive because A and C can both occur if the die comes up 2.
e) Yes, there are three mutually exclusive events among A, B, C, and D, namely, B, C, and D; indeed, no two of those three events can occur simultaneously. However, because A and B are not mutually exclusive, neither are A, B, C, and D.

f) Event $\{5\}$ is that the die comes up 5; event $\{1, 3, 5\}$ is that the die comes up odd; event $\{1, 2, 3, 4\}$ is that the die comes up at most 4.

2.5
a) The outcome of the random experiment is the tree obtained. Hence, the sample space consists of the set of 39 trees.
b) Event B^c is that the tree obtained does not have at least 20% seed damage; in other words, that it has less than 20% seed damage. This event is comprised of $19 + 2 = 21$ trees.
c) Event $C \cap D$ is that the tree obtained has at least 30% but less than 60% seed damage and that it has at least 50% seed damage; in other words, that it has at least 50% but less than 60% seed damage. This event is comprised of two trees.
d) Event $A \cup D$ is that the tree obtained has either less than 40% seed damage or at least 50% seed damage. This event is comprised of $19 + 2 + 5 + 3 + 2 + 2 = 33$ trees.
e) Event C^c is that the tree obtained does not have at least 30% but less than 60% seed damage; in other words, that it has either less than 30% seed damage or at least 60% seed damage. This event is comprised of $19 + 2 + 5 + 2 = 28$ trees.
f) Event $A \cap D$ is that the tree obtained has less than 40% seed damage and at least 50% seed damage, which is impossible. This event is comprised of zero trees.
g) We note that $A \cap B \neq \emptyset$, $A \cap C \neq \emptyset$, $A \cap D = \emptyset$, $B \cap C \neq \emptyset$, $B \cap D \neq \emptyset$, and $C \cap D \neq \emptyset$. Therefore, events A and D are mutually exclusive, whereas no other two events among A, B, C, and D are mutually exclusive. Moreover, because any collection of three or four events among A, B, C, and D must contain two events that do not consist of A and D, we deduce that no such collection consists of mutually exclusive events.

2.6
a) One possible sample space for this random experiment is

$$\Omega = \left\{ (x, y, z) : x, y, z \in \{0, 1, 2, \ldots, 9\} \text{ and } x \neq y \neq z \right\},$$

where x, y, and z represent the numbers on the first, second, and third balls selected, respectively.
b) Let E denote the event that an even number of odd-numbered balls are removed from the urn. Then event E occurs if and only if either no odd-numbered balls are removed or exactly two odd-numbered balls are removed. Using the sample space specified in the solution to part (a), we can express this event as

$$E = \left\{ (x, y, z) \in \Omega : \text{either } x, y, \text{ and } z \text{ are all even or exactly one of } x, y, \text{ and } z \text{ is even} \right\}.$$

2.7 We have $A_i = \left\{ (x, y) : x, y \in \{1, 2, 3, 4, 5, 6\} \text{ and } x + y = i \right\}$ for $i = 2, 3, \ldots, 12$. Hence,

$$\begin{aligned}
A_2 &= \{(1, 1)\} \\
A_3 &= \{(1, 2), (2, 1)\} \\
A_4 &= \{(1, 3), (2, 2), (3, 1)\} \\
A_5 &= \{(1, 4), (2, 3), (3, 2), (4, 1)\} \\
A_6 &= \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} \\
A_7 &= \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} \\
A_8 &= \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\} \\
A_9 &= \{(3, 6), (4, 5), (5, 4), (6, 3)\} \\
A_{10} &= \{(4, 6), (5, 5), (6, 4)\} \\
A_{11} &= \{(5, 6), (6, 5)\} \\
A_{12} &= \{(6, 6)\}
\end{aligned}$$

2.8

a) A typical outcome of this random experiment can be represented as an ordered pair of integers, (x, y), where x is the number of dots facing up on the die and y is the number of heads obtained when the coin is tossed x times. Hence, a sample space for this random experiment is

$$\Omega = \left\{ (x, y) : x \in \{1, 2, \dots, 6\} \text{ and } y \in \{0, 1, \dots, x\} \right\}.$$

b) Let E denote the event that the total number of heads is even. Then

$$\begin{aligned} E &= \{ (x, y) \in \Omega : y \text{ is even and } y \leq x \} \\ &= \{(1,0), (2,0), (2,2), (3,0), (3,2), (4,0), (4,2), (4,4), (5,0), (5,2), (5,4), \\ &\quad (6,0), (6,2), (6,4), (6,6)\}. \end{aligned}$$

2.9

a) A typical outcome of this random experiment can be represented as a string of heads followed by a single tail. Hence, a sample space for this random experiment is $\Omega = \{\text{T, HT, HHT, HHHT}, \dots\}$.

b) Let E denote the event that Laura wins. We note that E occurs if and only if the first tail occurs on an even-numbered toss. Hence, we have that $E = \{\text{HT, HHHT, HHHHHT}, \dots\}$.

2.10

a) A typical outcome of this random experiment can be represented as a string of tails followed by a single head. Hence, a sample space for this random experiment is $\Omega = \{\text{H, TH, TTH, TTTH}, \dots\}$.

b) A typical outcome of this random experiment can be represented as a string of tails followed by a head, followed by another string of tails, followed by a single head. Hence, a sample space for this random experiment is $\Omega = \{\text{HH, HTH, THH, HTTH, THTH, TTHH}, \dots\}$.

c) Let E denote the event that the coin is tossed exactly six times. We note that E occurs if and only if the first five tosses are tails and the sixth toss is a head. Hence, we see that $E = \{\text{TTTTTH}\}$.

d) Let F denote the event that the coin is tossed exactly six times. We note that F occurs if and only if exactly one head occurs among the first five tosses and the sixth toss is a head. Hence, we see that $F = \{\text{HTTTTH,THTTTH,TTHTTH,TTTHTH,TTTTHH,}\}$.

2.11

a) Because there are 10 men and 8 women, and the jury is to consist of 12 people, the number of men chosen must be at least 4 and at most 10. Consequently, the sample space for this random experiment is $\Omega = \{4, 5, \dots, 10\}$.

b) We note that at least half of the 12 jurors are men if and only if at least six men are on the jury; so, $A = \{6, 7, 8, 9, 10\}$. Also, at least half of the eight women are on the jury if and only if at least four women are on the jury, which is the case if and only if at most eight men are on the jury; so, $B = \{4, 5, 6, 7, 8\}$. We therefore see that

$$A \cup B = \{4, 5, \dots, 10\} = \Omega, \qquad A \cap B = \{6, 7, 8\}, \qquad \text{and} \qquad A \cap B^c = \{9, 10\}.$$

c) From part (b), we see that $A \cap B = \{6, 7, 8\} \neq \emptyset$; thus, A and B are not mutually exclusive. Also from part (b), we have that $A \cap B^c = \{9, 10\} \neq \emptyset$; thus, A and B^c are not mutually exclusive. However, we have $A^c \cap B^c = \{4, 5\} \cap \{9, 10\} = \emptyset$ and, thus, A^c and B^c are mutually exclusive.

2.12

a) Let $\omega \in A$. As A and B^c are mutually exclusive, we must have $\omega \notin B^c$, that is, $\omega \in B$. Hence, $A \subset B$, which, from Proposition 2.1 on page 32, means that B occurs whenever A occurs.

b) If B occurs whenever A occurs, then, from Proposition 2.1, we have $A \subset B$. Now, let $\omega \in A$. Then $\omega \in B$, so that $\omega \notin B^c$. Thus, A and B^c are mutually exclusive.

2.13

a) Event A occurs but event B doesn't occur if and only if the outcome, ω, of the random experiment is a member of A but not of B, which means that $\omega \in A$ and $\omega \in B^c$; that is, $\omega \in A \cap B^c$. Thus, the event that A occurs but B doesn't occur is $A \cap B^c$.

b) Exactly one of A and B occurs if and only if either A occurs but B doesn't occur or B occurs but A doesn't occur. Referring to part (a), we conclude that the event that exactly one of A and B occurs is $(A \cap B^c) \cup (A^c \cup B)$.

c) We first note that, for events E, F, and G, event E occurs but events F and G don't occur if and only if event E occurs but event $F \cup G$ doesn't occur. From part (a), then, the event that E occurs but F and G don't occur is

$$E \cap (F \cup G)^c = E \cap \left(F^c \cap G^c\right) = E \cap F^c \cap G^c.$$

Now, exactly one of A, B, and C occurs if and only if A occurs but B and C don't occur or B occurs but A and C don't occur or C occurs but A and B don't occur. Therefore, the event that exactly one of A, B, and C occurs is $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$ or, equivalently,

$$\left(A \cap B^c \cap C^c\right) \cup \left(A^c \cap B \cap C^c\right) \cup \left(A^c \cap B^c \cap C\right).$$

d) At most two of the events A, B, and C occur if and only if not all three of the events occur, which is the event $(A \cap B \cap C)^c$ or, equivalently, $A^c \cup B^c \cup C^c$.

2.14

a) A typical outcome of this random experiment can be represented as an ordered pair (x, y), where x and y are each one of the 52 cards, but are different. Here, x and y denote the first and second cards selected, respectively. Thus, letting S denote the collection of the 52 cards, a sample space for this random experiment is $\Omega = \{ (x, y) : x, y \in S \text{ and } x \neq y \}$.

b) No, events A and B are not mutually exclusive. For instance, both events occur if the first card selected is the king of spades and the second card selected is the ace of spades.

2.15

a) We can reasonably presume that patients cannot arrive simultaneously. Thus, the time of the fourth patient to arrive must be strictly after 6:00 P.M. Consequently, a typical outcome of this random experiment is some positive real number, representing the elapsed time, in hours, from 6:00 P.M. until the arrival of the fourth patient. In other words, a sample space for this random experiment is $(0, \infty)$.

b) A typical outcome of this random experiment can be represented by a nonnegative integer, which indicates the number of patients who arrive during the first half hour. Hence, a sample space for this random experiment is $\{0, 1, 2, \ldots\}$.

c) The event, B, that the fourth patient arrives before 6:30 P.M. occurs if and only if at least four patients arrive before 6:30 P.M. Event A occurs if and only if at least five patients arrive before 6:30 P.M., which implies that at least four patients arrive before 6:30 P.M., that is, that event B occurs. Hence, event B occurs whenever event A occurs, which, by Proposition 2.1 on page 32, means that $A \subset B$.

d) A sample space for this random experiment is $\{ (t_1, t_2, t_3, \ldots) : 0 < t_1 < t_2 < t_3 < \cdots \}$, where t_j denotes the elapsed time in hours from 6:00 P.M. until the arrival of the jth patient who arrives after 6:00 P.M.

As a preliminary to the solutions of parts (e)–(m), we establish some fundamental relationships among the events A, B, and C. We begin by noting that the event A is that the first nine patients don't have sprained ankles, which happens if and only if Smith leaves upon or after the arrival of the tenth patient, which happens if and only if Smith leaves after or at the same time as Jones, which occurs if and only if Jones leaves no later than Smith, which is event C. Hence, we see that $A = C$. Furthermore, we note that both events B and C (or A) occur if and only if Smith leaves no later than Jones and Jones leaves no later than Smith, which happens if and only if Smith and Jones leave at the same time, which we call event D. Thus, we have

$$A \cap B = B \cap C = D \neq \emptyset.$$

In addition, we note that D is a proper subset of each of A, B, and C.

e) False. Events A and B are not mutually exclusive because $A \cap B = D \neq \emptyset$.

f) False. Events A and C are not mutually exclusive because $A \cap C = C \cap C = C \neq \emptyset$.

g) False. Events B and C are not mutually exclusive because $B \cap C = D \neq \emptyset$.

h) False. If A occurs whenever B occurs, then we would have $B \subset A$ and, hence, $A \cap B = B$. But, then, we would have $B = D$, which, as we have noted, is not the case.

i) False. If B occurs whenever A occurs, then we would have $A \subset B$ and, hence, $A \cap B = A$. But, then, we would have $A = D$, which, as we have noted, is not the case.

j) True. A occurs whenever C occurs because $A = C$.

k) True. C occurs whenever A occurs because $A = C$.

l) False. If B occurs whenever C occurs, then we would have $C \subset B$ and, hence, $C \cap B = C$. But, then, we would have $C = D$, which, as we have noted, is not the case.

m) False. If C occurs whenever B occurs, then we would have $B \subset C$ and, hence, $B \cap C = B$. But, then, we would have $B = D$, which, as we have noted, is not the case.

2.16 Set $B_n = A_{n+1} \cap A_n^c$ for each $n \in \mathcal{N}$. We want to show that the events $B_1, B_2, \ldots$ are (pairwise) mutually exclusive. Now, let $m, n \in \mathcal{N}$ with $m \neq n$, say, $m < n$. Then $m + 1 \leq n$ and, hence, $A_{m+1} \subset A_n$. Consequently,

$$B_m \cap B_n = \left(A_{m+1} \cap A_m^c\right) \cap \left(A_{n+1} \cap A_n^c\right) \subset A_{m+1} \cap A_n^c \subset A_n \cap A_n^c = \emptyset.$$

Thus, $B_m \cap B_n = \emptyset$. Hence, we have shown that $A_2 \cap A_1^c, A_3 \cap A_2^c, A_4 \cap A_3^c, \ldots$ are mutually exclusive.

Advanced Exercises

2.17

a) If $\Omega = \{a, b\}$, then the events are $\emptyset$, $\{a\}$, $\{b\}$, and Ω.

b) If $\Omega = \{a, b, c\}$, then the events are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and Ω.

c) If $\Omega = \{a, b, c, d\}$, then the events are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and Ω.

d) From parts (a)–(c), we see that, if Ω has exactly two, three, or four elements, then it has 4, 8, and 16 events, respectively. We therefore claim that, if Ω has exactly n elements, then it has 2^n events. To prove that result, we use mathematical induction. From part (a), the result is true for $n = 2$. Assuming its truth for $n - 1$, we prove it for n. So, suppose that Ω has exactly n elements, say, $\Omega = \{a_1, \ldots, a_n\}$. The number of events of Ω that do not contain a_n equals the total number of events of $\Omega' = \{a_1, \ldots, a_{n-1}\}$, which, by the induction assumption is 2^{n-1}. By adding a_n to each of those 2^{n-1} events, we obtain all the events of Ω that do contain a_n. Hence, the total number of events of Ω is $2^{n-1} + 2^{n-1} = 2^n$, as required.

2.18

a) For each $n \in \mathcal{N}$, we have $\bigcup_{k=n}^{\infty} A_k \supset A_n \cup A_{n+1}$ and, hence,

$$A^* = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right) \supset \bigcap_{n=1}^{\infty}(A_n \cup A_{n+1}).$$

If $A_2, A_4, A_6, A_8, \ldots$ occur and $A_1, A_3, A_5, A_7, \ldots$ fail to occur, then, in particular, events $A_2, A_4, A_6, A_8, \ldots$ occur. Hence, the outcome, ω, of the random experiment is an element of A_{2k} for all $k \in \mathcal{N}$, which, in turn, implies that $\omega \in A_n \cup A_{n+1}$ for all $n \in \mathcal{N}$ and, thus, $\omega \in \bigcap_{n=1}^{\infty}(A_n \cup A_{n+1}) \subset A^*$. Consequently, if $A_2, A_4, A_6, A_8, \ldots$ occur and $A_1, A_3, A_5, A_7, \ldots$ fail to occur, then A^* occurs.

b) This result is obtained by using an argument entirely similar to the one used in the solution to part (a).
c) For each $n \in \mathcal{N}$, we have $10^n \geq n$ and, so $\bigcup_{k=n}^{\infty} A_k \supset A_{10^n}$. Hence,

$$A^* = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right) \supset \bigcap_{n=1}^{\infty} A_{10^n}.$$

If $A_{10}, A_{100}, A_{1000}, \ldots, A_{10^n}, \ldots$ occur and A_n fails to occur for any other value of n, then, in particular, events $A_{10}, A_{100}, A_{1000}, \ldots, A_{10^n}, \ldots$ occur. Hence, the outcome, ω, of the random experiment is an element of A_{10^n} for all $n \in \mathcal{N}$, which, in turn, implies that $\omega \in \bigcap_{n=1}^{\infty} A_{10^n} \subset A^*$. Consequently, if $A_{10}, A_{100}, A_{1000}, \ldots, A_{10^n}, \ldots$ occur and A_n fails to occur for any other value of n, then A^* occurs.
d) If A_{86}, A_{2049}, and $A_{30498541}$ occur and A_n fails to occur for each positive integer n such that $n \notin \{86, 2049, 30498541\}$, then, in particular, A_n fails to occur for all $n \geq 30498542$. Hence, the outcome, ω, of the random experiment is not an element of A_n for all $n \geq 30498542$, which, in turn implies that $\omega \notin \bigcup_{k=30498542}^{\infty} A_k$. As A^* is a subset of this latter set, we conclude that $\omega \notin A^*$, that is, A^* fails to occur.
e) We note that $\omega \in A^*$ if and only if $\omega \in \bigcup_{k=n}^{\infty} A_k$ for all $n \in \mathcal{N}$, which is the case if and only if for each $n \in \mathcal{N}$, there is a $k \geq n$ such that $\omega \in A_k$. Thus, A^* consists of all elements of Ω that belong to an infinite number of the A_ns. In other words, event A^* occurs if and only if $\{\, n \in \mathcal{N} : A_n \text{ occurs} \,\}$ is infinite. We interpret this fact by saying that A^* is the event that infinitely many of the A_ns occur.

2.19
a) Suppose that conditions (1) and (2) of a σ-algebra hold. Let $A_n \in \mathcal{A}$ for $n = 1, 2, \ldots$. Then, from condition (1), we have $A_n^c \in \mathcal{A}$ for all $n = 1, 2, \ldots$, which implies, by condition (2), that $\bigcap_n A_n^c \in \mathcal{A}$ and, hence, by condition (1) again, that $\left(\bigcap_n A_n^c\right)^c \in \mathcal{A}$. However, from De Morgan's laws,

$$\bigcup_n A_n = \bigcup_n (A_n^c)^c = \left(\bigcap_n A_n^c\right)^c.$$

Therefore, we have $\bigcup_n A_n \in \mathcal{A}$. Thus, condition (3) of a σ-algebra holds.
b) Noting that, here, Ω is the universal set, we see, from the definitions of complement, intersection, and union, that applying those three set operations to subsets of Ω results in other subsets of Ω. Therefore, the collection of all subsets of Ω is a σ-algebra.
c) From part (a), it suffices to show that conditions (1) and (2) of a σ-algebra hold. We see that those two conditions are satisfied by noting that $\Omega^c = \emptyset$, $\emptyset^c = \Omega$, and $\Omega \cap \emptyset = \emptyset$.
d) We note that applying the set operations of complement, intersection, and union to the sets Ω, E, E^c, and $\emptyset$ can only yield one of those four sets. Hence, the collection of those four sets form a σ-algebra.
e) We begin by noting that an argument similar to the one used in part (a) shows that conditions (1) and (3) of a σ-algebra imply condition (2). Let $A \in \mathcal{D}$. Then there is an $I \subset \{1, 2, \ldots\}$ such that $A = \bigcup_{n \in I} E_n$. Because the E_ns are mutually exclusive and have union Ω,

$$A^c = \left(\bigcup_{n \in I} E_n\right)^c = \bigcap_{n \in I} E_n^c = \bigcap_{n \in I}\left(\bigcup_{k \neq n} E_k\right) = \bigcup_{n \in I^c} E_n.$$

Thus, $A^c \in \mathcal{D}$; therefore, condition (1) holds. Now suppose that $A_1, A_2, \ldots$ are members of $\mathcal{D}$. Then there are subsets $I_1, I_2, \ldots$ of $\{1, 2, \ldots\}$ such that $A_k = \bigcup_{n \in I_k} E_n$ for all k. Consequently,

$$\bigcup_k A_k = \bigcup_k \left(\bigcup_{n \in I_k} E_n\right) = \bigcup_{n \in I} E_n,$$

where $I = \bigcup_k I_k$. Thus, $\bigcup_k A_k \in \mathcal{D}$; therefore, condition (3) holds.

f) No, $\mathcal{D}$ is not a σ-algebra. Indeed, because Ω is infinite, we can choose an infinite sequence of distinct elements of Ω, say, $\omega_1, \omega_2, \ldots$. Let $A_n = \{\omega_{2n}\}$ for each $n \in \mathcal{N}$. Then, as each A_n is finite, we have $A_n \in \mathcal{D}$ for all $n \in \mathcal{N}$. We will show that $\bigcup_{n=1}^{\infty} A_n \notin \mathcal{D}$, thus establishing that $\mathcal{D}$ is not a σ-algebra. Let $A = \bigcup_{n=1}^{\infty} A_n = \{\omega_2, \omega_4, \ldots\}$. We see that A is infinite and, because $A^c \supset \{\omega_1, \omega_3, \ldots\}$, we also see that A^c is infinite. Thus, neither A nor A^c is finite, which means that $A \notin \mathcal{D}$, that is, that $\bigcup_{n=1}^{\infty} A_n \notin \mathcal{D}$.

2.2 Axioms of Probability

Basic Exercises

2.20 We use the following table:

Location	Outcome ω	Probability $P(\{\omega\})$
Atlantic Ocean	ω_1	0.011
Pacific Ocean	ω_2	0.059
Gulf of Mexico	ω_3	0.271
Great Lakes	ω_4	0.018
Other lakes	ω_5	0.003
Rivers and canals	ω_6	0.211
Bays and sounds	ω_7	0.094
Harbors	ω_8	0.099
Other	ω_9	0.234

a) In view of Proposition 2.3 on page 43, we see that the probability assignment presented in the preceding table is legitimate from a probabilistic point of view because the numbers in the third column of the table are nonnegative and sum to 1.

Note: For parts (b)–(d), we refer to the preceding table and to Proposition 2.2 on page 42.

b) Let E denote the event that an oil spill (in U.S. navigable and territorial waters) occurs in an ocean. Then $E = \{\omega_1, \omega_2\}$ and, hence,

$$P(E) = \sum_{\omega \in E} P(\{\omega\}) = P(\{\omega_1\}) + P(\{\omega_2\}) = 0.011 + 0.059 = 0.070.$$

c) Let F denote the event that an oil spill occurs in a lake or harbor. Then $F = \{\omega_4, \omega_5, \omega_8\}$ and, hence,

$$P(F) = \sum_{\omega \in F} P(\{\omega\}) = P(\{\omega_4\}) + P(\{\omega_5\}) + P(\{\omega_8\}) = 0.018 + 0.003 + 0.099 = 0.120.$$

d) Let G denote the event that an oil spill doesn't occur in a lake, ocean, river, or canal. Then we have that $G = \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6\}^c = \{\omega_3, \omega_7, \omega_8, \omega_9\}$ and, hence,

$$\begin{aligned} P(G) &= \sum_{\omega \in G} P(\{\omega\}) = P(\{\omega_3\}) + P(\{\omega_7\}) + P(\{\omega_8\}) + P(\{\omega_9\}) \\ &= 0.271 + 0.094 + 0.099 + 0.234 = 0.698. \end{aligned}$$

2.21 As noted, here $\Omega = \{\text{H}, \text{T}\}$.

a) The numbers p and $1 - p$ are nonnegative and sum to 1. Hence, by Proposition 2.3 on page 43, there is a unique probability measure on the events of Ω such that $P(\{\text{H}\}) = p$ and $P(\{\text{T}\}) = 1 - p$.

b) The four events of this random experiment are $\emptyset$, $\{\text{H}\}$, $\{\text{T}\}$, and Ω. We know that $P(\{\text{H}\}) = p$ and $P(\{\text{T}\}) = 1 - p$. Moreover, for any probability measure, we have $P(\emptyset) = 0$ and $P(\Omega) = 1$.

2.22

a) In view of Proposition 2.3 on page 43, we need only check each assignment for nonnegativity and summing to 1.

Assignment #1: As $1/6 \geq 0$ and $1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 1$, we see that assignment #1 is a legitimate probability assignment.

Assignment #2: As the six numbers are nonnegative and sum to 1, we see that assignment #2 is a legitimate probability assignment.

Assignment #3: We have $0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 = 1.2 \neq 1$. Hence, assignment #3 is not a legitimate probability assignment.

Assignment #4: Because $-1/4 < 0$, assignment #4 is not a legitimate probability assignment.

Assignment #5: As the six numbers are nonnegative and sum to 1, we see that assignment #5 is a legitimate probability assignment.

b) We have $A = \{2, 4, 6\}$, $B = \{4, 5, 6\}$, $C = \{1, 2\}$, and $D = \{3\}$. To obtain the probabilities of these four events, we apply Proposition 2.2 on page 42.

Assignment #1: We have

$$\begin{aligned}
P(A) &= P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/6 + 1/6 + 1/6 = 1/2, \\
P(B) &= P(\{4\}) + P(\{5\}) + P(\{6\}) = 1/6 + 1/6 + 1/6 = 1/2, \\
P(C) &= P(\{1\}) + P(\{2\}) = 1/6 + 1/6 = 1/3, \\
P(D) &= P(\{3\}) = 1/6.
\end{aligned}$$

Assignment #2: We have

$$\begin{aligned}
P(A) &= P(\{2\}) + P(\{4\}) + P(\{6\}) = 0.15 + 0.05 + 0.20 = 0.4, \\
P(B) &= P(\{4\}) + P(\{5\}) + P(\{6\}) = 0.05 + 0.10 + 0.20 = 0.35, \\
P(C) &= P(\{1\}) + P(\{2\}) = 0.10 + 0.15 = 0.25, \\
P(D) &= P(\{3\}) = 0.4.
\end{aligned}$$

Assignment #5: We have

$$\begin{aligned}
P(A) &= P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/8 + 0 + 1/8 = 1/4, \\
P(B) &= P(\{4\}) + P(\{5\}) + P(\{6\}) = 0 + 7/16 + 1/8 = 9/16, \\
P(C) &= P(\{1\}) + P(\{2\}) = 1/16 + 1/8 = 3/16, \\
P(D) &= P(\{3\}) = 1/4.
\end{aligned}$$

c) Let D denote the event that the die comes up odd. Then $D = \{1, 3, 5\}$ and we have

$$\begin{aligned}
P(D) = P(\{1\}) + P(\{3\}) + P(\{5\}) &= \begin{cases} 1/6 + 1/6 + 1/6, & \text{for assignment \#1;} \\ 0.10 + 0.40 + 0.10, & \text{for assignment \#2;} \\ 1/16 + 1/4 + 7/16, & \text{for assignment \#5.} \end{cases} \\
&= \begin{cases} 1/2, & \text{for assignment \#1;} \\ 0.6, & \text{for assignment \#2;} \\ 3/4, & \text{for assignment \#5.} \end{cases}
\end{aligned}$$

d) If the die is balanced, then each of the six possible outcomes are equally likely. Hence, assignment #1 should be used.

e) In view of the frequentist interpretation of probability on page 5, we see that, in this case, assignment #2 should be used.

2.23

a) In view of Proposition 2.3 on page 43, we need only check each assignment for nonnegativity and summing to 1.

Assignment #1: We have $1/2^n \geq 0$ for all $n \in \mathcal{N}$ and, from the formula for a geometric series,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1 - 1/2} = 1.$$

Hence, assignment #1 is a legitimate probability assignment.

Assignment #2: As $1/3 \geq 0$ and $1/3 + 1/3 + 1/3 + 0 + 0 + \cdots = 1$, we see that assignment #2 is a legitimate probability assignment.

Assignment #3: From the study of infinite series, we know that $\sum_{n=2}^{\infty} 1/n = \infty \neq 1$. Hence, assignment #3 is not a legitimate probability assignment.

Assignment #4: As $1 \geq 0$ and $0 + 0 + 1 + 0 + 0 + 0 + \cdots = 1$, we see that assignment #4 is a legitimate probability assignment.

Assignment #5: As $0 < p \leq 1$, we have $p(1-p)^n \geq 0$ for all $n \in \mathcal{N}$. Also, from the formula for a geometric series,

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \cdot \frac{1}{1-(1-p)} = 1.$$

Hence, assignment #5 is a legitimate probability assignment.

b) Let A_n denote the event that a success occurs by the nth attempt. Then A_n consists of the first n outcomes in the first column of the table presented in the problem statement. To obtain the required probabilities, we apply Proposition 2.2 on page 42.

Assignment #1: We have

$$\begin{aligned}
P(A_2) &= 1/2 + 1/4 = 3/4, \\
P(A_4) &= 1/2 + 1/4 + 1/8 + 1/16 = 15/16, \\
P(A_6) &= 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 = 63/64.
\end{aligned}$$

Assignment #2: We have

$$\begin{aligned}
P(A_2) &= 1/3 + 1/3 = 2/3, \\
P(A_4) &= 1/3 + 1/3 + 1/3 + 0 = 1, \\
P(A_6) &= 1/3 + 1/3 + 1/3 + 0 + 0 + 0 = 1.
\end{aligned}$$

Assignment #4: We have

$$\begin{aligned}
P(A_2) &= 0 + 0 = 0, \\
P(A_4) &= 0 + 0 + 1 + 0 = 1, \\
P(A_6) &= 0 + 0 + 1 + 0 + 0 + 0 = 1.
\end{aligned}$$

Assignment #5: We begin here by noting that

$$\sum_{k=1}^{n} p(1-p)^{k-1} = p \sum_{k=1}^{n} (1-p)^{k-1} = p \cdot \frac{1-(1-p)^n}{1-(1-p)} = 1 - (1-p)^n.$$

Therefore, we have

$$P(A_2) = 1 - (1-p)^2,$$
$$P(A_4) = 1 - (1-p)^4,$$
$$P(A_6) = 1 - (1-p)^6.$$

c) No, if $p = 0$, then assignment #5 is not a legitimate probability assignment. Indeed, in that case, we have

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = \sum_{n=1}^{\infty} 0 \cdot 1^{n-1} = \sum_{n=1}^{\infty} 0 = 0 \neq 1.$$

In terms of the random experiment, $p = 0$ means that a success is impossible. To obtain a legitimate probability assignment in that case, add the outcome $f, f, f, \ldots$ (an unending string of fs) to the sample space and assign it probability 1.

2.24

a) Applying Exercise 1.34 with $A_1 = A$ and $A_2 = A^c$, we deduce that $B = (B \cap A) \cup (B \cap A^c)$, where the sets in the union are disjoint (i.e., $B \cap A$ and $B \cap A^c$ are mutually exclusive). Hence, by the additivity axiom for a probability measure,

$$P(B) = P\big((B \cap A) \cup (B \cap A^c)\big) = P(B \cap A) + P(B \cap A^c).$$

b) Using properties of set operations and referring to the solution to part (a) with B replaced by $A \cup B$, we deduce that

$$\begin{aligned} A \cup B &= \big((A \cup B) \cap A\big) \cup \big((A \cup B) \cap A^c\big) = A \cup \big((A \cap A^c) \cup (B \cap A^c)\big) \\ &= A \cup \emptyset \cup (B \cap A^c) = A \cup (B \cap A^c), \end{aligned}$$

where the events in the last union are mutually exclusive. Hence, by the additivity axiom for a probability measure,

$$P(A \cup B) = P\big(A \cup (B \cap A^c)\big) = P(A) + P(B \cap A^c).$$

c) Applying in turn the certainty axiom for a probability measure, part (b), and part (a), we get

$$1 = P(\Omega) = P(A \cup B) = P(A) + P(B \cap A^c) = P(A) + \big(P(B) - P(A \cap B)\big).$$

Hence, $P(A \cap B) = P(A) + P(B) - 1$.

2.25

a) We have

$$1/36 \geq 0 \qquad \text{and} \qquad \underbrace{1/36 + \cdots + 1/36}_{36 \text{ times}} = 1.$$

Hence, from Proposition 2.3 on page 43, the specified probability assignment is legitimate.

b) Let E be an event of this random experiment. Applying Proposition 2.2 on page 42 with the probability assignment from part (a), we get that

$$P(E) = \sum_{\omega \in E} P(\{\omega\}) = \sum_{\omega \in E} \frac{1}{36} = \frac{N(E)}{36},$$

where $N(E)$ denotes the number of outcomes that comprise event E.

By referring to the solution to Exercise 2.7, we then find that

$$P(A_2) = P(\{(1,1)\}) = \frac{1}{36}$$
$$P(A_3) = P(\{(1,2),(2,1)\}) = \frac{2}{36}$$
$$P(A_4) = P(\{(1,3),(2,2),(3,1)\}) = \frac{3}{36}$$
$$P(A_5) = P(\{(1,4),(2,3),(3,2),(4,1)\}) = \frac{4}{36}$$
$$P(A_6) = P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) = \frac{5}{36}$$
$$P(A_7) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36}$$
$$P(A_8) = P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{5}{36}$$
$$P(A_9) = P(\{(3,6),(4,5),(5,4),(6,3)\}) = \frac{4}{36}$$
$$P(A_{10}) = P(\{(4,6),(5,5),(6,4)\}) = \frac{3}{36}$$
$$P(A_{11}) = P(\{(5,6),(6,5)\}) = \frac{2}{36}$$
$$P(A_{12}) = P(\{(6,6)\}) = \frac{1}{36}$$

More concisely, we have

$$P(A_i) = \begin{cases} (i-1)/36, & 2 \le i \le 7; \\ (13-i)/36, & 8 \le i \le 12. \end{cases}$$

c) Answers will vary. In view of Proposition 2.2, any assignment of 36 numbers to the 36 possible outcomes will work provided that those numbers are nonnegative and sum to 1. To obtain the probabilities of the A_is using your probability assignment, apply Proposition 2.2.
d) Assuming the die is balanced, the 36 possible outcomes are equally likely and, hence, the only proper assignment of probabilities is the one presented in part (a). In particular, then, the assignment in part (c), being different from the one in part (a), is not reasonable.

2.26
a) We have

$$0.01 \ge 0 \qquad \text{and} \qquad \underbrace{0.01 + \cdots + 0.01}_{100 \text{ times}} = 1.$$

Hence, from Proposition 2.3 on page 43, the specified probability assignment is legitimate.
b) Let E be an event of this random experiment. Applying Proposition 2.2 on page 42 with the probability assignment from part (a), we get that

$$P(E) = \sum_{\omega \in E} P(\{\omega\}) = \sum_{\omega \in E} 0.01 = 0.01N(E),$$

where $N(E)$ denotes the number of outcomes that comprise event E. Now, we have $A = \{2, 4, \ldots, 100\}$ and, hence,

$$P(A) = 0.01N(A) = 0.01 \cdot 50 = 0.5.$$

Also, as $10\pi = 31.415\ldots$, we have $B = \{1, 2, \ldots, 31\}$ and, hence,

$$P(B) = 0.01N(B) = 0.01 \cdot 31 = 0.31.$$

Furthermore,

$$C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$$

and, hence,

$$P(C) = 0.01N(C) = 0.01 \cdot 25 = 0.25.$$

2.27

a) A typical outcome of this random experiment can be represented as an ordered pair of integers between 1 and 4, inclusive, where the first and second integers are the numbers on the first and second balls chosen, respectively. Hence, a sample space for this random experiment is

$$\Omega = \left\{ (x, y) : x, y \in \{1, 2, 3, 4\} \right\}.$$

b) Let p denote the common probability. Applying Corollary 2.1 on page 43 and noting that Ω has 16 members, we get

$$1 = \sum_{\omega \in \Omega} P(\{\omega\}) = \sum_{\omega \in \Omega} p = 16p.$$

Hence, $p = 1/16$.

c) Let D denote the event that the two numbers chosen are different. As there are 16 total possible outcomes and 4 outcomes in which the two numbers chosen are the same [namely, (1, 1), (2, 2), (3, 3), and (4, 4)], we see that there are 12 outcomes in which the two numbers chosen are different. Hence, from Proposition 2.2 on page 42 and part (b),

$$P(D) = \sum_{\omega \in D} P(\{\omega\}) = \sum_{\omega \in D} 1/16 = 12 \cdot 1/16 = 3/4.$$

2.28

a) Because the N possible outcomes are equally likely, each must have the same probability. Let p denote the common probability. Applying Corollary 2.1 on page 43, we get

$$1 = \sum_{\omega \in \Omega} P(\{\omega\}) = \sum_{\omega \in \Omega} p = Np.$$

Hence, $p = 1/N$.

b) Let E be an event consisting of m outcomes. Applying Proposition 2.2 on page 42 with the probability assignment from part (a), we get that

$$P(E) = \sum_{\omega \in E} P(\{\omega\}) = \sum_{\omega \in E} 1/N = m \cdot 1/N = m/N.$$

Theory Exercises

2.29

a) If $P(A_i) = 1/6$ for each i and $n = 10$, then Relation $(*)$ tells us that

$$P(A_1 \cup \cdots \cup A_{10}) \le P(A_1) + \cdots + P(A_{10}) = \underbrace{1/6 + \cdots + 1/6}_{10 \text{ times}} = \frac{5}{3},$$

that is, that the union of the 10 events has probability at most $5/3$. This result is trivial because the probability of any event must be at most 1.

If $P(A_i) = 1/60$ for each i and $n = 10$, then Relation $(*)$ tells us that

$$P(A_1 \cup \cdots \cup A_{10}) \le P(A_1) + \cdots + P(A_{10}) = \underbrace{1/60 + \cdots + 1/60}_{10 \text{ times}} = \frac{1}{6},$$

that is, that the union of the 10 events has probability at most $1/6$. This result certainly conveys some useful information. Likewise, if $P(A_i) = 1/6$ for each i and $n = 4$, then Relation $(*)$ tells us that

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) \le P(A_1) + P(A_2) + P(A_3) + P(A_4) = 1/6 + 1/6 + 1/6 + 1/6 = \frac{2}{3},$$

that is, that the union of the four events has probability at most $2/3$. This result also conveys some useful information.

b) We use mathematical induction to prove Relation $(*)$. For $n = 1$, it states that $P(A_1) \le P(A_1)$, which is certainly true. Assuming its truth for $n-1$, we prove it for n. For convenience, set $B_n = \bigcup_{i=1}^{n-1} A_i$. From the induction assumption, we have

$$P(B_n) = P\left(\bigcup_{i=1}^{n-1} A_i\right) \le \sum_{i=1}^{n-1} P(A_i).$$

Now we apply, in turn, Exercises 2.24(b) and 2.24(a) with $A = B_n$ and $B = A_n$, and the nonnegativity axiom to deduce that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(B_n \cup A_n) = P(B_n) + P(A_n \cap B_n^c) = P(B_n) + P(A_n) - P(A_n \cap B_n)$$

$$\le P(B_n) + P(A_n) \le \sum_{i=1}^{n-1} P(A_i) + P(A_n) = \sum_{i=1}^{n} P(A_i),$$

as required.

Advanced Exercises

2.30 For convenience, set $\mathcal{N}_0 = \{0, 1, 2, \ldots\}$.

a) Clearly, $e^{-5}\, 2^i 3^j / i!\, j! \ge 0$ for all $(i, j) \in \mathcal{N}_0 \times \mathcal{N}_0$. Furthermore,

$$\sum\sum_{(i,j)\in\mathcal{N}_0\times\mathcal{N}_0} e^{-5}\,\frac{2^i 3^j}{i!\, j!} = e^{-2}e^{-3}\left(\sum_{i=0}^{\infty} \frac{2^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{3^j}{j!}\right) = e^{-2}e^{-3}e^{2}e^{3} = 1,$$

where, in the penultimate equality, we applied the exponential series, $e^t = \sum_{k=0}^{\infty} t^k/k!$. Hence, in view of Proposition 2.3 on page 43, the specified probability assignment is legitimate.

b) Let A denote the event that the point selected at random is on the x-axis. Then $A = \{\, (i, 0) : i \in \mathcal{N}_0 \,\}$. Applying Proposition 2.2 on page 42, we get

$$P(A) = \sum_{i=0}^{\infty} e^{-5}\,\frac{2^i 3^0}{i!\, 0!} = e^{-5}\sum_{i=0}^{\infty} \frac{2^i}{i!} = e^{-5}e^{2} = e^{-3}.$$

c) Let B denote the event that the point selected at random is in the closed square of side length 3 with lower left vertex at the origin. Then $B = \{0, 1, 2, 3\} \times \{0, 1, 2, 3\}$ and, hence, from Proposition 2.2,

$$P(B) = \sum\sum_{(i,j)\in\{0,1,2,3\}\times\{0,1,2,3\}} e^{-5}\,\frac{2^i 3^j}{i!\, j!} = e^{-5}\left(\sum_{i=0}^{3} \frac{2^i}{i!}\right)\left(\sum_{j=0}^{3} \frac{3^j}{j!}\right) = e^{-5} \cdot \frac{19}{3} \cdot 13 = \frac{247}{3}e^{-5}.$$

d) Let C denote the event that the point selected at random has x coordinate i, where i is a nonnegative integer. Then $C = \{\,(i, j) : j \in \mathcal{N}_0\,\}$. Applying Proposition 2.2, we get

$$P(C) = \sum_{j=0}^{\infty} e^{-5}\,\frac{2^i 3^j}{i!\,j!} = e^{-5}\,\frac{2^i}{i!}\sum_{j=0}^{\infty}\frac{3^j}{j!} = e^{-5}\,\frac{2^i}{i!}\,e^3 = e^{-2}\,\frac{2^i}{i!}.$$

e) Let D denote the event that the point selected at random has y coordinate j, where j is a nonnegative integer. Then $D = \{\,(i, j) : i \in \mathcal{N}_0\,\}$. Applying Proposition 2.2, we get

$$P(D) = \sum_{i=0}^{\infty} e^{-5}\,\frac{2^i 3^j}{i!\,j!} = e^{-5}\,\frac{3^j}{j!}\sum_{i=0}^{\infty}\frac{2^i}{i!} = e^{-5}\,\frac{3^j}{j!}\,e^2 = e^{-3}\,\frac{3^j}{j!}.$$

2.31

a) Because $p_{11}, p_{12}, p_{13}, \ldots$ is a legitimate probability assignment for Ω_1 and $p_{21}, p_{22}, p_{23}, \ldots$ is a legitimate probability assignment for Ω_2, we know that $p_{ij} \geq 0$ for $i \in \{1, 2\}$ and $j \in \mathcal{N}$ and, furthermore, that $\sum_{j=1}^{\infty} p_{ij} = 1$ for $i \in \{1, 2\}$. Therefore, we have $p_{1m}p_{2n} \geq 0$ for all $m, n \in \mathcal{N}$ and

$$\sum\sum_{(m,n)\in\mathcal{N}\times\mathcal{N}} p_{1m}p_{2n} = \left(\sum_{m=1}^{\infty} p_{1m}\right)\left(\sum_{n=1}^{\infty} p_{2n}\right) = 1 \cdot 1 = 1.$$

Hence, in view of Proposition 2.3 on page 43, the specified probability assignment for Ω is legitimate.

Note: In parts (b)–(d), we apply Proposition 2.2 on page 42.

b) We have

$$P\big(\{(\omega_{11}, \omega_{21}), (\omega_{12}, \omega_{24})\}\big) = P\big(\{(\omega_{11}, \omega_{21})\}\big) + P\big(\{(\omega_{12}, \omega_{24})\}\big) = p_{11}p_{21} + p_{12}p_{24}.$$

c) We have

$$P\big(\{\,(\omega_{1m}, \omega_{2n}) : n \in \mathcal{N}\,\}\big) = \sum_{n=1}^{\infty} P\big(\{(\omega_{1m}, \omega_{2n})\}\big) = \sum_{n=1}^{\infty} p_{1m}p_{2n} = p_{1m}\sum_{n=1}^{\infty} p_{2n} = p_{1m}\cdot 1 = p_{1m}.$$

d) We have

$$P\big(\{\,(\omega_{1m}, \omega_{2n}) : m \in \mathcal{N}\,\}\big) = \sum_{m=1}^{\infty} P\big(\{(\omega_{1m}, \omega_{2n})\}\big) = \sum_{m=1}^{\infty} p_{1m}p_{2n} = p_{2n}\sum_{m=1}^{\infty} p_{1m} = p_{2n}\cdot 1 = p_{2n}.$$

2.32 Answers will vary. One possibility is obtained as follows. Here the sample space consists of the rational numbers, $\mathcal{Q}$. Let $\{\omega_n\}_{n=1}^{\infty}$ be an enumeration of $\mathcal{Q}$ and assign probability 2^{-n} to outcome ω_n. We have $2^{-n} > 0$ for all $n \in \mathcal{N}$. Furthermore, from the geometric series,

$$\sum_{n=1}^{\infty} 2^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - 1/2} = 1.$$

Hence, from Proposition 2.3 on page 43, the specified probability assignment is legitimate.

2.33 Here the sample space is the unit square: $\Omega = [0, 1] \times [0, 1]$. We have

$$P(\{\omega\}) = \text{Area}(\{\omega\}) = 0, \qquad \omega \in \Omega.$$

We claim that Ω is uncountable. Suppose to the contrary that Ω is countable. Applying Corollary 2.1, we then conclude that

$$1 = \sum_{\omega\in\Omega} P(\{\omega\}) = \sum_{\omega\in\Omega} 0 = 0,$$

a contradiction. Hence, Ω is uncountable.

2.34 *Note:* In parts (a)–(c), we let x denote the price, in dollars, that you are willing to pay to be promised that, if the Red Sox win next year's World Series, you will receive \$1.

a) If you set $x > 1$, your opponent can always beat you by selling the promise to you (for \$$x$). Indeed, in that case, if the Red Sox win, your opponent's net gain will be \$$(x - 1) >$ \$0, whereas, if the Red Sox lose, your opponent's net gain will be \$$(x - 0) >$ \$1. Thus, whether the Red Sox win or lose, your opponent will have a positive net gain. Hence, setting a price higher than \$1 is bad for you.

b) If you set $x < 0$, your opponent can always beat you by buying the promise from you (for \$$x$). Indeed, in that case, if the Red Sox win, your opponent's net gain will be \$$(1 - x) >$ \$1, whereas, if the Red Sox lose, your opponent's net gain will be \$$(0 - x) >$ \$0. Thus, whether the Red Sox win or lose, your opponent will have a positive net gain. Hence, setting a price lower than \$0 is bad for you.

c) On the one hand, suppose that you are absolutely certain that the Red Sox will win. If you then set a price lower than \$1 (i.e., $x < 1$), you would fear that your opponent would buy the promise from you because, in that case, you would lose \$$(1 - x) >$ \$0. If, however, you set a price of \$1 (i.e., $x = 1$), you would not fear that your opponent would buy the promise from you because, in that case, you would break even.

On the other hand, suppose that you are absolutely certain that the Red Sox will not win. If you then set a price higher than \$0 (i.e., $x > 0$), you would fear that your opponent would sell the promise to you because, in that case, you would lose \$$(x - 0)$. If, however, you set a price of \$0 (i.e., $x = 0$), you would not fear that your opponent would sell the promise to you because, in that case, you would break even.

Note: In parts (d) and (e), we let x, y, and z denote the prices, in dollars, that you are willing to pay to be promised that you will receive \$1 if the Red Sox, Diamondbacks, and Red Sox or Diamondbacks, respectively, win next year's World Series.

d) If you set the prices so that $x + y > z$, your opponent can always beat you by selling you the promises for both the Red Sox and the Diamondbacks and buying the promise for the Red Sox or Diamondbacks. Indeed, in that case, if the Red Sox win, your opponent's net gain will be

$$(x - 1) + (y - 0) + (1 - z) = x + y - z > 0.$$

If the Diamondbacks win, your opponent's net gain will be

$$(x - 0) + (y - 1) + (1 - z) = x + y - z > 0.$$

And, if neither the Red Sox nor the Diamondbacks win, your opponent's net gain will be

$$(x - 0) + (y - 0) + (0 - z) = x + y - z > 0.$$

Thus, regardless of which team wins, your opponent will have a positive net gain. Hence, setting the prices so that $x + y > z$ is bad for you.

e) If you set the prices so that $x + y < z$, your opponent can always beat you by buying the promises for both the Red Sox and the Diamondbacks and selling you the promise for the Red Sox or Diamondbacks. Indeed, in that case, if the Red Sox win, your opponent's net gain will be

$$(1 - x) + (0 - y) + (z - 1) = z - x - y > 0.$$

If the Diamondbacks win, your opponent's net gain will be

$$(0 - x) + (1 - y) + (z - 1) = z - x - y > 0.$$

And, if neither the Red Sox nor the Diamondbacks win, your opponent's net gain will be

$$(0 - x) + (0 - y) + (z - 0) = z - x - y > 0.$$

Thus, regardless of which team wins, your opponent will have a positive net gain. Hence, setting the prices so that $x + y < z$ is bad for you.

2.3 Specifying Probabilities

Basic Exercises

2.35 Because the ball is equally likely to land on any of the 38 numbers, a classical probability model is appropriate here. So, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{38}.$$

a) Let R denote the event that the ball lands on a red number. As there are 18 red numbers, we have $N(R) = 18$. Hence,

$$P(R) = \frac{N(R)}{38} = \frac{18}{38} = \frac{9}{19} \approx 0.474.$$

b) Let B denote the event that the ball lands on a black number. As there are 18 black numbers, we have $N(B) = 18$. Hence,

$$P(B) = \frac{N(B)}{38} = \frac{18}{38} = \frac{9}{19} \approx 0.474.$$

c) Let G denote the event that the ball lands on a green number. As there are two green numbers, we have $N(G) = 2$. Hence,

$$P(G) = \frac{N(G)}{38} = \frac{2}{38} = \frac{1}{19} \approx 0.0526.$$

d) The event that the ball lands on either a black or green number is $B \cup G$. As B and G are mutually exclusive, we can apply the additivity axiom and the results of parts (b) and (c) to conclude that

$$P(B \cup G) = P(B) + P(G) = \frac{9}{19} + \frac{1}{19} = \frac{10}{19} \approx 0.526.$$

We note that this probability can also be obtained directly by observing that $N(B \cup G) = 20$.

e) The event that the number on which the ball lands is not black is B^c. Because there are $38 - 18 = 20$ non-black numbers, we have

$$P(B^c) = \frac{N(B^c)}{38} = \frac{20}{38} = \frac{10}{19} \approx 0.526.$$

f) The event that the number on which the ball lands is not green is G^c. Because there are $38 - 2 = 36$ non-green numbers, we have

$$P(G^c) = \frac{N(G^c)}{38} = \frac{36}{38} = \frac{18}{19} \approx 0.947.$$

2.36

a) An equal-likelihood model would be appropriate if and only if an observed component is equally likely to be working or to have failed; in other words, if and only if the probabilility is 1/2 that an observed component is working.

b) Presuming that an equal-likelood model is appropriate, we have, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{32}.$$

c) Let A denote the event that the number of nonworking components is at most one. Then we have $A = \{(s, s, s, s, s), (s, s, s, s, f), (s, s, s, f, s), (s, s, f, s, s), (s, f, s, s, s), (f, s, s, s, s)\}$. Hence,

$$P(A) = \frac{N(A)}{32} = \frac{6}{32} = \frac{3}{16}.$$

d) Let B denote the event that the number of nonworking components is at least one. We note that B occurs if and only if the outcome of the random experiment is not (s, s, s, s, s). Hence,

$$P(B) = \frac{N(B)}{32} = \frac{32 - 1}{32} = \frac{31}{32}.$$

e) Let C denote the event that the number of nonworking components is zero. Then $C = \{(s, s, s, s, s)\}$ and, hence,

$$P(C) = \frac{N(C)}{32} = \frac{1}{32}.$$

2.37
a) An equal-likelihood model would be appropriate if and only if the six genotypes (*aa,ab,ao,bb,bo,oo*) are equally likely.
b) Let B denote the event that the person chosen has type B blood. Referring to Table 2.2, we see that $B = \{bb, bo\}$. Hence, assuming an equal-likelihood model, we have

$$P(B) = \frac{N(B)}{N(\Omega)} = \frac{2}{6} = \frac{1}{3}.$$

c) No, an equal-likelihood model is not appropriate because, as is well known, genotypes are not equally likely.
d) As we noted in part (b), we have $B = \{bb, bo\}$. Applying Proposition 2.2 on page 42 and referring to the table provided, we deduce that

$$P(B) = P(\{bb\}) + P(\{bo\}) = 0.007 + 0.116 = 0.123,$$

which is significantly smaller than the (incorrect) probability obtained in part (b).

2.38 Denote by $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ the events labeling the rows and columns, respectively, of the contingency table. We note that events $A_1, \ldots, A_m$ are mutually exclusive, as are events $B_1, \ldots, B_n$. The joint events in the contingency table are $A_i \cap B_j$, where $1 \le i \le m$ and $1 \le j \le n$. Now, suppose that $(i, j) \ne (k, \ell)$. Then either $i \ne k$ or $j \ne \ell$ (or both). Hence, either $A_i \cap A_k = \emptyset$ or $B_j \cap B_\ell = \emptyset$. Consequently,

$$(A_i \cap B_j) \cap (A_k \cap B_\ell) = (A_i \cap A_k) \cap (B_j \cap B_\ell) = \emptyset.$$

Thus, the joint events in a contingency table are mutually exclusive.

2.39
a) This contingency table has $3 \times 4 = 12$ cells.
b) We see from the lower right corner of the contingency table that there are (were) 94 players on the New England Patriots roster as of May 24, 2000.
c) From the last entry in the first column of the contingency table (the first entry in the "Total" row), we see that the number of rookies is 42.
d) From the last entry in the second row of the contingency table (the second entry in the "Total" column), we see that the number of players who weigh between 200 and 300 lb is 54.
e) From the cell in the second row and first column of the contingency table, we see that the number of players who are rookies and weigh between 200 and 300 lb is 22.
f) Event Y_3 is that the player chosen has between 6 and 10 years of experience, inclusive; event W_2 is that the player chosen weighs between 200 and 300 lb, inclusive; event $W_1 \cap Y_2$ is that the player chosen weighs under 200 lb and has between 1 and 5 years of experience, inclusive.

g) Because a player is selected at random, each of the 94 players is equally likely to be the one chosen. Consequently, a classical probability model is appropriate here. So, for each event E, we have

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{94}.$$

Thus,

$$P(Y_3) = \frac{N(Y_3)}{94} = \frac{8}{94} = \frac{4}{47} \approx 0.0851,$$

$$P(W_2) = \frac{N(W_2)}{94} = \frac{54}{94} = \frac{27}{47} \approx 0.574,$$

and

$$P(W_1 \cap Y_2) = \frac{N(W_1 \cap Y_2)}{94} = \frac{9}{94} \approx 0.0957.$$

h) Proceeding as in part (h), we obtain the following joint probability distribution:

Years of experience

Weight (lb)	Rookie Y_1	1–5 Y_2	6–10 Y_3	10+ Y_4	$P(W_i)$
Under 200 W_1	0.096	0.096	0.021	0.011	0.223
200–300 W_2	0.234	0.266	0.053	0.021	0.574
Over 300 W_3	0.117	0.074	0.011	0.000	0.202
$P(Y_j)$	0.447	0.436	0.085	0.032	1.000

i) For the rows, we have

$$0.096 + 0.096 + 0.021 + 0.011 = 0.224 \text{ (s/b 0.223)}$$
$$0.234 + 0.266 + 0.053 + 0.021 = 0.574$$
$$0.117 + 0.074 + 0.011 + 0.000 = 0.202$$

and, for the columns,

$$0.096 + 0.234 + 0.117 = 0.447$$
$$0.096 + 0.266 + 0.074 = 0.436$$
$$0.021 + 0.053 + 0.011 = 0.085$$
$$0.011 + 0.021 + 0.000 = 0.032.$$

2.40

a) Yes. As the dice are balanced, each of the 36 possible outcomes is equally likely to occur. Hence, a classical probability model is appropriate here. So, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{36}.$$

b) Let A denote the event that the sum of the dice (two numbers) is 4. Then $A = \{(1,3), (2,2), (3,1)\}$ and, hence,

$$P(A) = \frac{N(A)}{36} = \frac{3}{36} = \frac{1}{12}.$$

c) Let B denote the event that the sum of the dice is 5. Then $B = \{(1,4), (2,3), (3,2), (4,1)\}$ and, hence,

$$P(B) = \frac{N(B)}{36} = \frac{4}{36} = \frac{1}{9}.$$

d) Let C denote the event that the minimum of the two numbers (showing on the dice) is 4. Then we see that $C = \{(4,4), (4,5), (4,6), (5,4), (6,4)\}$ and, hence,

$$P(C) = \frac{N(C)}{36} = \frac{5}{36}.$$

e) Let D denote the event that the maximum of the two numbers (showing on the dice) is 4. Then we see that $D = \{(1,4), (2,4), (3,4), (4,4), (4,1), (4,2), (4,3)\}$ and, hence,

$$P(D) = \frac{N(D)}{36} = \frac{7}{36}.$$

2.41 Let us denote the four senators by r1, r2, d1, and d2, where r and d stand for Republican and Democrat, respectively. A sample space for the random experiment of selecting two of the four senators to constitute a subcommittee is $\Omega = \big\{\{\text{r1}, \text{r2}\}, \{\text{r1}, \text{d1}\}, \{\text{r1}, \text{d2}\}, \{\text{r2}, \text{d1}\}, \{\text{r2}, \text{d2}\}, \{\text{d1}, \text{d2}\}\big\}$. Because the selection is done at random, each of the six possible outcomes is equally likely. Thus, a classical probability model is appropriate here. So, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

Now, for $i = 0$, 1, and 2, let A_i denote the event that the number of Republicans on the subcommittee is exactly i.

a) We have $A_2 = \big\{\{\text{r1}, \text{r2}\}\big\}$ and, hence,

$$P(A_2) = \frac{N(A_2)}{6} = \frac{1}{6} \approx 0.167.$$

b) We have $A_1 = \big\{\{\text{r1}, \text{d1}\}, \{\text{r1}, \text{d2}\}, \{\text{r2}, \text{d1}\}, \{\text{r2}, \text{d2}\}\big\}$ and, hence,

$$P(A_1) = \frac{N(A_1)}{6} = \frac{4}{6} = \frac{2}{3} \approx 0.667.$$

c) We have $A_0 = \big\{\{\text{d1}, \text{d2}\}\big\}$ and, hence,

$$P(A_0) = \frac{N(A_0)}{6} = \frac{1}{6} \approx 0.167.$$

2.42 John will wait between 0 and 30 minutes for the train. Hence, a sample space for the random experiment is $\Omega = [0, 30)$, where the outcome represents the number of minutes that John waits for the train. As John casually strolls to the train station, we can reasonably presume that a geometric probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\big|[0, 30)\big|} = \frac{|E|}{30},$$

where $|E|$ denotes the length of the set E.

a) Let A denote the event that John waits for the train between 10 and 15 minutes. Then $A = [10, 15]$ and, hence,

$$P(A) = \frac{|A|}{30} = \frac{15 - 10}{30} = \frac{5}{30} = \frac{1}{6} \approx 0.167.$$

b) Let B denote the event that John waits for the train at least 10 minutes. Then $B = [10, 30)$ and, hence,

$$P(B) = \frac{|B|}{30} = \frac{30 - 10}{30} = \frac{20}{30} = \frac{2}{3} \approx 0.667.$$

2.43 As we discovered earlier, we can take the sample space for this random experiment to be the unit disk, $\Omega = \{ (x, y) : x^2 + y^2 < 1 \}$, and, moreover, we can reasonably use a geometric probability model. Thus, we can think of the location of (the center of) the first spot (visible bacteria colony) as a point selected at random from the unit disk. Consequently, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\pi} = \frac{1}{\pi}|E|,$$

where $|E|$ denotes the area of the set E.

a) Let A denote the event that the distance from the center (of the disk) to the first spot is less than half the distance from the center to the rim. Then $A = \{ (x, y) : \sqrt{x^2 + y^2} < 1/2 \}$ and, hence,

$$P(A) = \frac{1}{\pi}|A| = \frac{1}{\pi} \cdot \pi \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25.$$

b) Let B denote the event that the distance from the center to the first spot is more than half the distance from the center to the rim. Then $B = \{ (x, y) : \sqrt{x^2 + y^2} > 1/2 \}$ and, hence,

$$P(B) = \frac{1}{\pi}|B| = \frac{1}{\pi} \cdot \left(\pi - \pi \cdot \left(\frac{1}{2}\right)^2\right) = \frac{3}{4} = 0.75.$$

c) Let C denote the event that the center of the first spot to appear is not in the inscribed square centered at the origin with vertices on the circle $x^2 + y^2 = 1$ and sides parallel to the coordinate axes. Noting that the forementioned square has side of length $\sqrt{2}$, we get

$$P(C) = \frac{1}{\pi}|C| = \frac{1}{\pi} \cdot \left(\pi - \sqrt{2} \cdot \sqrt{2}\right) = 1 - \frac{2}{\pi} \approx 0.363.$$

d) Let D be the event that the center of the first spot to appear is on the strip $\{ (x, y) : -1/4 < x < 1/4 \}$, which we denote S. Noting that $D = \Omega \cap S$, we have

$$P(D) = \frac{1}{\pi}|D| = \frac{1}{\pi}|\Omega \cap S|.$$

Now, let T denote the first-quadrant portion of $\Omega \cap S$. Applying calculus, we find that

$$P(D) = \frac{1}{\pi}|\Omega \cap S| = \frac{4}{\pi}|T| = \frac{4}{\pi}\int_0^{1/4} \sqrt{1 - x^2}\, dx = \frac{4}{\pi} \cdot \frac{1}{2}\left[x\sqrt{1 - x^2} + \arcsin x\right]_0^{1/4}$$

$$= \frac{2}{\pi} \cdot \left(\frac{1}{4}\sqrt{1 - \left(\frac{1}{4}\right)^2} + \arcsin(1/4)\right) = \frac{\sqrt{15}}{8\pi} + \frac{2}{\pi}\arcsin(1/4) \approx 0.315.$$

2.44 Because a point is chosen at random from the unit square, a geometric probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\left|[0,1]^2\right|} = \frac{|E|}{1 \cdot 1} = \frac{|E|}{1} = |E|,$$

where $|E|$ denotes the area of the set E. *Note:* In the solution to this exercise and the next, when we refer to a geometric object, such as a rectangle or a triangle, we mean the interior of that object, possible containing part or all of the boundary of the object.

a) Event A is that the point chosen has x coordinate greater than 1/3. We note that A is the rectangle with vertices $(1/3, 0)$, $(1, 0)$, $(1, 1)$, and $(1/3, 1)$, which has area $(1 - 1/3) \cdot (1 - 0) = 2/3$. Hence, we have $P(A) = |A| = 2/3$.

b) Event B is that the point chosen has y coordinate at most 0.7. We note that B is the rectangle with vertices $(0, 0)$, $(1, 0)$, $(1, 0.7)$, and $(0, 0.7)$, which has area $(1 - 0) \cdot (0.7 - 0) = 0.7$. Hence, we have $P(B) = |B| = 0.7$.

c) Event C is that the sum of the two coordinates of the point chosen exceeds 1.2. We note that C is the triangle with vertices $(0.2, 1)$, $(1, 1)$, and $(1, 0.2)$, which has area $\big((1 - 0.2) \cdot (1 - 0.2)\big)/2 = 0.32$. Hence, we have $P(C) = |C| = 0.32$.

d) Event D is that the magnitude of the difference of the two coordinates of the point chosen is less than 0.1. We note that event D^c consists of two triangles, one with vertices $(0.1, 0)$, $(1, 0)$ and $(1, 0.9)$ and the other with vertices $(0, 0.1)$, $(0, 1)$ and $(0.9, 1)$. Therefore,

$$P(D) = |D| = |\Omega| - |D^c| = 1 - 2 \cdot \frac{1}{2} \cdot (0.9)^2 = 1 - 0.81 = 0.19.$$

e) Event E is that the two coordinates of the point chosen are equal. We note that E is the diagonal line segment from $(0, 0)$ to $(1, 1)$. Therefore, we have $P(E) = |E| = 0$.

2.45 From Exercise 2.44, we know that $P(E) = |E|$ for each event E, where $|E|$ denotes the area of the set E. Also, see the note at the beginning of the solution to that exercise.

a) Let A denote the event that the x coordinate of the point chosen is less than the y coordinate. We note that A is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. Hence, $P(A) = |A| = (1 \cdot 1)/2 = 1/2$.

b) Let B denote the event that the smaller of the two coordinates of the point chosen is less than 1/2. We note that event B^c is the rectangle with vertices $(1/2, 1/2)$, $(1/2, 1)$, $(1, 1)$, and $(1, 1/2)$. Hence, we have

$$P(B) = |B| = |\Omega| - |B^c| = 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

c) Let C denote the event that the smaller of the two coordinates of the point chosen exceeds 1/2. We note that C is the rectangle with vertices $(1/2, 1/2)$, $(1/2, 1)$, $(1, 1)$, and $(1, 1/2)$. Hence, we have $P(C) = |C| = (1/2) \cdot (1/2) = 1/4$.

d) Let D denote the event that the larger of the two coordinates of the point chosen is less than 1/2. We note that D is the rectangle with vertices $(0, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, and $(1/2, 0)$. Hence, we have $P(D) = |D| = (1/2) \cdot (1/2) = 1/4$.

e) Let E denote the event that the larger of the two coordinates of the point chosen exceeds 1/2. We note that event E^c is the rectangle with vertices $(0, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, and $(1/2, 0)$. Hence, we have

$$P(E) = |E| = |\Omega| - |E^c| = 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

f) Let F denote the event that the sum of the two coordinates is between 1 and 1.5. We note that event F^c consists of two triangles, one with vertices (0, 0), (0, 1), and (1, 0) and the other with vertices (1/2, 1), (1, 1), and (1, 1/2). Hence, we have

$$P(F) = |F| = |\Omega| - |F^c| = 1 - \left(\frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{3}{8}.$$

g) Let G denote the event that the sum of the two coordinates is between 1.5 and 2. We note that G is the triangle vertices (1/2, 1), (1, 1), and (1, 1/2). Hence, $P(G) = |G| = (1/2) \cdot (1/2) \cdot (1/2) = 1/8$.

2.46 The sample space for this random experiment is $\Omega = (0, 1)$. Moreover, because a number is being chosen at random, a geometric probability model is appropriate. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{|(0, 1)|} = \frac{|E|}{1} = |E|,$$

where $|E|$ denotes the length of the set E.

a) Let A denote the event that the first digit of the decimal expansion of the number chosen is 7. We note that A is the interval $[0.7, 0.8)$. Hence, $P(A) = |A| = \big|[0.7, 0.8)\big| = 0.8 - 0.7 = 0.1$.

b) Let B denote the event that the second digit of the decimal expansion of the number chosen is 7. We note that $B = \bigcup_{k=0}^{9} I_k$, where $I_k = [0.k7, 0.k8)$. Hence, from the additivity axiom,

$$P(B) = P\left(\bigcup_{k=0}^{9} I_k\right) = \sum_{k=0}^{9} P(I_k) = \sum_{k=0}^{9} |I_k| = \sum_{k=0}^{9} 0.01 = 10 \cdot 0.1 = 0.1.$$

c) Let C denote the event that the second digit of the decimal expansion of the square root of the number chosen is 7, and let x be the number chosen. Then event C occurs if and only if $\sqrt{x} \in \bigcup_{k=0}^{9} I_k$, where $I_k = [0.k7, 0.k8)$. Now, $\sqrt{x} \in I_k$ if and only if $x \in J_k$, where $J_k = \big[(0.k7)^2, (0.k8)^2\big)$. We have

$$|J_k| = (0.k8)^2 - (0.k7)^2 = \left(\frac{k}{10} + \frac{8}{100}\right)^2 - \left(\frac{k}{10} + \frac{7}{100}\right)^2 = \frac{2k}{10^3} + \frac{15}{10^4}.$$

Consequently, we see that $C = \bigcup_{k=0}^{9} J_k$ and, by the additivity axiom,

$$\begin{aligned} P(C) = P\left(\bigcup_{k=0}^{9} J_k\right) &= \sum_{k=0}^{9} P(J_k) = \sum_{k=0}^{9} |J_k| = \sum_{k=0}^{9}\left(\frac{2k}{10^3} + \frac{15}{10^4}\right) \\ &= \frac{2}{10^3}\sum_{k=0}^{9} k + \frac{15}{10^4}\sum_{k=0}^{9} 1 = \frac{2}{10^3} \cdot 45 + \frac{15}{10^4} \cdot 10 = 0.105. \end{aligned}$$

2.47 The sample space, Ω, is the unit sphere. Because a point is being chosen at random, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{4\pi/3} = \frac{3}{4\pi}|E|,$$

where $|E|$ denotes the volume of the set E.

a) Let A denote the event that the point chosen is more than 1/4 unit from the origin. We note that event A^c is the sphere of radius 1/4 centered at the origin. Hence,

$$P(A) = \frac{3}{4\pi}|A| = \frac{3}{4\pi}\big(|\Omega| - |A^c|\big) = \frac{3}{4\pi}\left(\frac{4}{3}\pi - \frac{4}{3}\pi\left(\frac{1}{4}\right)^3\right) = \frac{63}{64} \approx 0.984.$$

b) Let B denote the event that the point chosen is in a cube of side length 1 centered at the origin. We note that the forementioned cube lies entirely within the unit cube. Hence,

$$P(B) = \frac{3}{4\pi}|B| = \frac{3}{4\pi} \cdot 1^3 = \frac{3}{4\pi} \approx 0.239.$$

c) Let C denote the event that the point chosen is on the surface of the sphere. The volume of the surface of the unit sphere is zero. Hence, $P(C) = 0$.

2.48 The sample space, Ω, for this random experiment is the interior of a triangle with base b and height h. As a point is selected at random from the interior of the triangle, a geometric probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{bh/2} = \frac{2}{bh}|E|,$$

where $|E|$ denotes the area of the set E.
a) Let A_x denote the event that the distance of the point selected from the base of the triangle is at most x. If $x < 0$, then $A_x = \emptyset$ and, hence, $P(A_x) = 0$. If $x \geq h$, then $A_x = \Omega$ and, hence, $P(A_x) = 1$. Now assume that $0 \leq x < h$. Then event A_x^c is a triangle of height $h - x$ and base $b(h - x)/h$. Hence,

$$\begin{aligned} P(A_x) &= \frac{2}{bh}|A_x| = \frac{2}{bh}\big(|\Omega| - |A_x^c|\big) = \frac{2}{bh}\left(\frac{1}{2}bh - \frac{1}{2} \cdot \frac{b(h-x)}{h} \cdot (h-x)\right) \\ &= 1 - \left(\frac{h-x}{h}\right)^2 = 1 - \left(1 - \frac{x}{h}\right)^2 = \frac{x}{h}\left(2 - \frac{x}{h}\right). \end{aligned}$$

Consequently,

$$P(A_x) = \begin{cases} 0, & \text{if } x < 0; \\ \dfrac{x}{h}\left(2 - \dfrac{x}{h}\right), & \text{if } 0 \leq x < h; \\ 1, & \text{if } x \geq h. \end{cases}$$

b) Let B_{xy} denote the event that the distance of the point selected from the base of the triangle is between x and y. We, of course, assume that $x < y$. Then $A_x \subset A_y$ and, hence, from the additivity axiom,

$$P(A_y) = P\big((A_y \cap A_x) \cup (A_y \cap A_x^c)\big) = P(A_y \cap A_x) + P(A_y \cap A_x^c) = P(A_x) + P(A_y \cap A_x^c).$$

Noting that $B_{xy} = A_y \cap A_x^c$, we therefore have $P(B_{xy}) = P(A_y \cap A_x^c) = P(A_y) - P(A_x)$. Observe that

$$\frac{y}{h}\left(2 - \frac{y}{h}\right) - \frac{x}{h}\left(2 - \frac{x}{h}\right) = \frac{y-x}{h}\left(2 - \frac{x+y}{h}\right).$$

Consequently, in view of part (a),

$$P(B_{xy}) = \begin{cases} 0, & \text{if } y < 0 \text{ or } x \geq h; \\ \dfrac{y}{h}\left(2 - \dfrac{y}{h}\right), & \text{if } x < 0 \text{ and } 0 \leq y < h; \\ \dfrac{y-x}{h}\left(2 - \dfrac{x+y}{h}\right), & \text{if } 0 \leq x < y < h; \\ 1 - \dfrac{x}{h}\left(2 - \dfrac{x}{h}\right), & \text{if } 0 \leq x < h \text{ and } y \geq h; \\ 1, & \text{if } x < 0 \text{ and } y \geq h. \end{cases}$$

2.49 For definiteness, we position the side of the triangle from which the point is selected on the horizontal axis and center it at the origin. Then a sample space for the random experiment is $\Omega = [-\ell/2, \ell/2]$ and, as the point is selected at random, a geometric probability model is appropriate. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\big|[-\ell/2, \ell/2]\big|} = \frac{|E|}{\ell},$$

where $|E|$ denotes the length of the set E.

a) Let A_x denote the event that the distance of the point selected from the opposite vertex is at most x. We note that the height of the triangle is $\sqrt{3}\ell/2$. If $x < \sqrt{3}\ell/2$, then $A_x = \emptyset$ and, hence, $P(A_x) = 0$. If $x \geq \ell$, then $A_x = \Omega$ and, hence, $P(A_x) = 1$. Now assume that $\sqrt{3}\ell/2 \leq x < \ell$. From the Pythagorean theorem, we find that A_x is the interval with endpoints $\pm\sqrt{4x^2 - 3\ell^2}/2$. Hence,

$$P(A_x) = \frac{|E|}{\ell} = \frac{\sqrt{4x^2 - 3\ell^2}}{\ell}.$$

Consequently,

$$P(A_x) = \begin{cases} 0, & \text{if } x < \sqrt{3}\ell/2; \\ \sqrt{4x^2 - 3\ell^2}\big/\ell, & \text{if } \sqrt{3}\ell/2 \leq x < \ell; \\ 1, & \text{if } x \geq \ell. \end{cases}$$

b) Let B_{xy} denote the event that the distance of the point selected from the opposite vertex is between x and y. We assume, of course, that $x < y$. Then $A_x \subset A_y$ and, hence, from the additivity axiom,

$$P(A_y) = P\big((A_y \cap A_x) \cup (A_y \cap A_x^c)\big) = P(A_y \cap A_x) + P(A_y \cap A_x^c) = P(A_x) + P(A_y \cap A_x^c).$$

Noting that $B_{xy} = A_y \cap A_x^c$, we conclude, in view of part (a), that

$$\begin{aligned} P(B_{xy}) &= P(A_y \cap A_x^c) = P(A_y) - P(A_x) \\ &= \begin{cases} 0, & \text{if } y < \sqrt{3}\ell/2 \text{ or } x \geq \ell; \\ \sqrt{4y^2 - 3\ell^2}\big/\ell, & \text{if } x < \sqrt{3}\ell/2 \text{ and } \sqrt{3}\ell/2 \leq y < \ell; \\ \left(\sqrt{4y^2 - 3\ell^2} - \sqrt{4x^2 - 3\ell^2}\right)\big/\ell, & \text{if } \sqrt{3}\ell/2 \leq x < y < \ell; \\ 1 - \sqrt{4x^2 - 3\ell^2}\big/\ell, & \text{if } \sqrt{3}\ell/2 \leq x < \ell \text{ and } y \geq \ell; \\ 1, & \text{if } x < \sqrt{3}\ell/2 \text{ and } y \geq \ell. \end{cases} \end{aligned}$$

2.50 The random experiment is observing the race (white, black, or other) of the mother of a newborn child in the United States. Thus, we can take the sample space to be $\Omega = \{w, b, o\}$, where w = white, b = black, and o = other.

a) For the given data, we are repeating the experiment 4,058,814 times. Based on the data, we have

$$P(\{w\}) \approx \frac{N(\{w\})}{n} = \frac{3{,}194{,}005}{4{,}058{,}814} \approx 0.787,$$

$$P(\{b\}) \approx \frac{N(\{b\})}{n} = \frac{622{,}598}{4{,}058{,}814} \approx 0.153,$$

$$P(\{o\}) \approx \frac{N(\{o\})}{n} = \frac{4{,}058{,}814 - 3{,}194{,}005 - 622{,}598}{4{,}058{,}814} = \frac{242{,}211}{4{,}058{,}814} \approx 0.0597.$$

b) The probabilities obtained in part (a) are empirical probabilities because they are based on the frequentist interpretation of probability.

2.51 The engineer is specifying a subjective probability, based on her previous limited interviewing experience, knowledge of the job market, and partial information about the company.

2.52 Answers will vary. We would say that the realtor was specifying a probability that was partially empirical (based on her previous experience in the resale housing market in Sun Lakes, Arizona) and partially subjective.

Advanced Exercises

2.53 A sample space for the random experiment is $\Omega = (0, 1)$. Because a number is being selected at random, a geometric probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{|(0, 1)|} = \frac{|E|}{1} = |E|,$$

where $|E|$ denotes the length of the set E. Let A denote the event that the number selected is rational. Then $A = \mathcal{Q} \cap (0, 1)$, which is a countable set, being a subset of the countable set $\mathcal{Q}$. Let $r_1, r_2, \ldots$ be an enumeration of A. Then, from the additivity axiom,

$$P(A) = P\left(\bigcup_{n=1}^{\infty} \{r_n\}\right) = \sum_{n=1}^{\infty} P(\{r_n\}) = \sum_{n=1}^{\infty} |\{r_n\}| = \sum_{n=1}^{\infty} (r_n - r_n) = \sum_{n=1}^{\infty} 0 = 0.$$

2.54 Suppose to the contrary that there is an equal-likelihood model for a countably infinite sample space, Ω. Then $P(\{\omega\})$ does not depend on ω, say, $P(\{\omega\}) = p$ for all $\omega \in \Omega$. Applying Corollary 2.1 on page 43, we get

$$1 = \sum_{\omega \in \Omega} P(\{\omega\}) = \sum_{\omega \in \Omega} p.$$

Because Ω is infinite, the sum on the right of the preceding display is an infinite sum and, hence, equals either 0 (if $p = 0$) or ∞ (if $p > 0$). In either case, we have a contradiction. Consequently, we have shown that having an equal-likelihood model for a countably infinite sample space isn't possible.

2.55 A sample space for this random experiment is

$$\Omega = (0, \ell)^3 = \{\,(x, y, z) : 0 < x < \ell,\ 0 < y < \ell,\ 0 < z < \ell\,\},$$

where x, y, and z represent the first, second, and third numbers selected, respectively. Because the three numbers are selected at random, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\left|(0, \ell)^3\right|} = \frac{|E|}{\ell^3},$$

where $|E|$ denotes the volume of the set E. Let A denote the event that a triangle can be formed from the three line segments whose lengths are the three numbers obtained. From geometry, we know that event A occurs if and only if each of the three numbers obtained is less than the sum of the other two numbers obtained. Hence, event A^c occurs if and only if one of the three numbers obtained is at least as large as the sum of the other two numbers obtained. Let B denote the event that the third number obtained is at least as large as the sum of the first two numbers obtained. Then $B = \{\,(x, y, z) \in \Omega : z \geq x + y\,\}$. By symmetry, we have $|A^c| = 3|B|$. Using calculus, we get

$$|B| = \iiint_B dx\,dy\,dz = \int_0^{\ell}\left(\int_0^{\ell - x}\left(\int_{x+y}^{\ell} dz\right) dy\right) dx = \frac{\ell^3}{6}.$$

Consequently,

$$P(A) = \frac{|A|}{\ell^3} = \frac{|\Omega| - |A^c|}{\ell^3} = \frac{\ell^3 - 3 \cdot \left(\ell^3/6\right)}{\ell^3} = \frac{1}{2}.$$

2.56 A sample space for this random experiment is

$$\Omega = (0, \ell)^2 = \{\, (x, y) : 0 < x < \ell,\ 0 < y < \ell \,\},$$

where x and y represent the first and second numbers selected, respectively. Because the two numbers are selected at random, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\left|(0, \ell)^2\right|} = \frac{|E|}{\ell^2},$$

where $|E|$ denotes the area of the set E. Let A denote the event that a triangle can be formed from the three line segments resulting from the two numbers selected. Also, let B denote the event that the x coordinate of the point selected is less than the y coordinate of the point selected. We have $A = (A \cap B) \cup (A \cap B^c)$ and, hence, from the additivity property and symmetry,

$$P(A) = P\big((A \cap B) \cup (A \cap B^c)\big) = P(A \cap B) + P(A \cap B^c) = 2P(A \cap B) = 2 \cdot \frac{|A \cap B|}{\ell^2}.$$

Now, event $A \cap B$ occurs if and only if $x < y$ and the length of each of the three line segments is less than the sum of the lengths of the other two line segments, which is the case if and only if $x < y$ and $x + (y - x) > \ell - y$, $x + (\ell - y) > y - x$, and $(y - x) + (\ell - y) > x$, which happens if and only if $x < y$, $y > \ell/2$, $y - x < \ell/2$, and $x < \ell/2$. Hence,

$$A \cap B = \{\, (x, y) \in \Omega : x < y,\ y > \ell/2,\ y - x < \ell/2, \text{ and } x < \ell/2 \,\}.$$

We thus see that $A \cap B$ is the (interior of the) triangle with vertices $(0, \ell/2)$, $(\ell/2, \ell)$ and $(\ell/2, \ell/2)$, which has area $\ell^2/8$. Consequently,

$$P(A) = 2 \cdot \frac{|A \cap B|}{\ell^2} = 2 \cdot \frac{\ell^2/8}{\ell^2} = \frac{1}{4}.$$

2.57 Let x denote the distance from the center of the needle to the closest line and let y denote the angle, in radians, which the needle forms with that line. We note that the quantities x and y completely determine the position of the needle. Hence, a sample space for the experiment of dropping the needle onto the floor is $\Omega = \{\, (x, y) : 0 \le x \le d/2,\ 0 \le y \le \pi \,\}$. As the needle is dropped randomly, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\pi d/2} = \frac{2}{\pi d}|E|,$$

where $|E|$ denotes the area of the set E. Let A denote the event that the needle crosses one of the lines. We see that event A occurs if and only if $x \le (\ell/2)\sin y$. Hence, $A = \{\, (x, y) \in \Omega : x \le (\ell/2)\sin y \,\}$. Using calculus, we get

$$|A| = \iint\limits_A dx\, dy = \int_0^\pi \left(\int_0^{(\ell/2)\sin y} dx \right) dy = \frac{\ell}{2} \int_0^\pi \sin y\, dy = \frac{\ell}{2} \cdot 2 = \ell.$$

Consequently,

$$P(A) = \frac{2}{\pi d}|A| = \frac{2}{\pi d} \cdot \ell = \frac{2\ell}{\pi d}.$$

2.4 Basic Properties of Probability

Basic Exercises

2.58 Answers will vary. Here is one possibility. Suppose that a point is selected at random from the interval (0, 1). As we have seen, for each event E, we have $P(E) = |E|$, where $|E|$ denotes the length of the set E. Let $A = (0, 1/3)$ and $B = (0, 1/3]$. Then A is a proper subset of B, but $P(A) = 1/3 = P(B)$.

2.59 Answers will vary. Here is one possibility. Consider a classical probability model on the sample space $\Omega = \{1, 2, 3\}$. Let $A = \{1\}$ and $B = \{2, 3\}$. Then

$$P(A) = P(\{1\}) = \frac{1}{3} \leq \frac{2}{3} = \frac{1}{3} + \frac{1}{3} = P(\{2\}) + P(\{3\}) = P(B),$$

but $A \not\subset B$.

2.60 It is more (or at least as) probable that the person chosen is a lawyer. Indeed, let L and R denote the events that the person chosen is a lawyer and a Republican, respectively. The event that the person chosen is a Republican lawyer is $L \cap R$. As $L \cap R \subset L$, the domination principle implies that $P(L \cap R) \leq P(L)$. Strict inequality occurs if and only if there is lawyer in the state who is not a Republican.

2.61 From Exercise 2.41, we know that choosing two senators at random from the four senators on the committee to form a subcommittee has six equally-likely outcomes. Let E denote the event that at least one Republican is on the subcommittee. We note that E^c is the event that no Republicans are on the subcommittee, that is, $E^c = \big\{\{\text{d1}, \text{d2}\}\big\}$. Hence, from the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - P\Big(\big\{\{\text{d1}, \text{d2}\}\big\}\Big) = 1 - \frac{1}{6} = \frac{5}{6} \approx 0.833.$$

By using the complementation rule, we need only obtain the probability of a simple event, whereas, if that rule were not used, we would have to obtain the probability of a compound event, that is, an event consisting of more than one outcome.

2.62 Let E denote the event that an oil spill in U.S. navigable and territorial waters doesn't occur in the Gulf of Mexico.

a) Referring to the table in the solution to Exercise 2.20, we see that $E = \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\}$ and, hence, that

$$\begin{aligned} P(E) &= \sum_{\omega \in E} P(\{\omega\}) \\ &= P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_4\}) + P(\{\omega_5\}) + P(\{\omega_6\}) + P(\{\omega_7\}) + P(\{\omega_8\}) + P(\{\omega_9\}) \\ &= 0.011 + 0.059 + 0.018 + 0.003 + 0.211 + 0.094 + 0.099 + 0.234 \\ &= 0.729. \end{aligned}$$

b) Event E^c is that it is not the case that an oil spill in U.S. navigable and territorial waters doesn't occur in the Gulf of Mexico, that is, that it does occur in the Gulf of Mexico. Referring to the table in the solution to Exercise 2.20, we see that $E^c = \{\omega_3\}$ and, hence, by the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - P(\{\omega_3\}) = 1 - 0.271 = 0.729.$$

c) Referring to the solutions in parts (a) and (b), we see that, using the complementation rule, as done in part (b), is a far easier way to obtain the required probability.

2.63 We refer to the table presented in the statement of Exercise 2.39. The sample space consists of the 94 players and, because a player is selected at random, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{94}.$$

a) Let A denote the event that the player chosen has at least 1 year of experience. Then A^c is the event that the player chosen is a rookie. Hence, from the complementation rule,

$$P(A) = 1 - P(A^c) = 1 - P(Y_1) = 1 - \frac{42}{94} = \frac{52}{94} \approx 0.553.$$

b) Let B denote the event that the player chosen weighs at most 300 lb. Then B^c is the event that the player chosen weighs more than 300 lb. Hence, from the complementation rule,

$$P(B) = 1 - P(B^c) = 1 - P(W_3) = 1 - \frac{19}{94} = \frac{75}{94} \approx 0.798.$$

c) From the general addition rule,

$$P(Y_1 \cup W_3) = P(Y_1) + P(W_3) - P(Y_1 \cap W_3) = \frac{42}{94} + \frac{19}{94} - \frac{11}{94} = \frac{50}{94} \approx 0.532.$$

Alternatively, we have

$$P(Y_1 \cup W_3) = \frac{9 + 22 + 11 + 7 + 1 + 0}{94} = \frac{50}{94} \approx 0.532.$$

2.64 The sample space, Ω, is the unit square and, because a point is selected at random, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{1} = |E|,$$

where $|E|$ denotes the area of the set E. Let A denote the event that the magnitude of the difference between the x and y coordinates of the point obtained is at most 1/4. Then $A = \{ (x, y) \in \Omega : |y - x| \leq 1/4 \}$. We note that $A^c = \{ (x, y) \in \Omega : |y - x| > 1/4 \}$, which is the union of two disjoint triangles, one with vertices (0, 1/4), (0, 1), and (3/4, 1) and the other with vertices (1/4, 0), (1, 0), and (1, 3/4). Hence,

$$|A^c| = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}.$$

From the complementation rule,

$$P(A) = 1 - P(A^c) = 1 - |A^c| = 1 - \frac{9}{16} = \frac{7}{16} = 0.4375.$$

Alternatively, we have

$$P(A) = |A| = |\Omega| - |A^c| = 1 - \frac{9}{16} = \frac{7}{16} = 0.4375.$$

2.65 For a randomly selected U.S. adult, let F and D denote the events that the person obtained is a female and is divorced, respectively. We know that $P(F) = 0.510$, $P(D) = 0.071$, and $P(F \cap D) = 0.041$.
a) From the general addition rule,

$$P(F \cup D) = P(F) + P(D) - P(F \cap D) = 0.510 + 0.071 - 0.041 = 0.540.$$

b) Let M denote the event that the person obtained is a male. Applying the complementation rule, we get

$$P(M) = 1 - P(M^c) = 1 - P(F) = 1 - 0.510 = 0.490.$$

c) We want to determine $P(F \cap D^c)$. Applying the law of partitions, we get

$$0.510 = P(F) = P(D \cap F) + P(D^c \cap F) = 0.041 + P(D^c \cap F) = 0.041 + P(F \cap D^c).$$

Hence, $P(F \cap D^c) = 0.510 - 0.041 = 0.469$.
d) We want to determine $P(M \cap D)$. Applying the law of partitions, we get

$$0.071 = P(D) = P(F \cap D) + P(F^c \cap D) = 0.041 + P(M \cap D).$$

Hence, $P(M \cap D) = 0.071 - 0.041 = 0.030$.

2.66
a) We have

$$P(A) + P(B) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12} \neq \frac{1}{2} = P(A \cup B).$$

Hence, A and B are not mutually exclusive.
b) From the general addition rule,

$$\frac{1}{2} = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{3} - P(A \cap B) = \frac{7}{12} - P(A \cap B).$$

Hence, $P(A \cap B) = 7/12 - 1/2 = 1/12$.

2.67
a) From the general addition rule,

$$\frac{5}{8} = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + P(B) - \frac{1}{10} = P(B) + \frac{7}{30}.$$

Hence, $P(B) = 5/8 - 7/30 = 47/120$.
b) From the law of partitions,

$$\frac{1}{3} = P(A) = P(B \cap A) + P(B^c \cap A) = P(A \cap B) + P(A \cap B^c) = \frac{1}{10} + P(A \cap B^c).$$

Hence, $P(A \cap B^c) = 1/3 - 1/10 = 7/30$.
c) From the general addition rule, the complementation rule, and parts (a) and (b),

$$\begin{aligned} P(A \cup B^c) &= P(A) + P(B^c) - P(A \cap B^c) = P(A) + 1 - P(B) - P(A \cap B^c) \\ &= \frac{1}{3} + 1 - 47/120 - 7/30 = \frac{17}{24}. \end{aligned}$$

d) From De Morgan's law and the complementation rule,

$$P\big(A^c \cup B^c\big) = P\big((A \cap B)^c\big) = 1 - P(A \cap B) = 1 - \frac{1}{10} = \frac{9}{10}.$$

2.68 Suppose that one of the 1200 people interviewed is selected at random. Let A and B denote the events that the person obtained enjoys his/her job and enjoys his/her personal life, respectively. We know that $P(A^c \cap B^c) = 0.15$, $P(A \cap B^c) = 0.80$, and $P(A \cap B) = 0.04$.
a) From De Morgan's law and the complementation rule,

$$P(A \cup B) = P\big((A^c \cap B^c)^c\big) = 1 - P\big(A^c \cap B^c\big) = 1 - 0.15 = 0.85.$$

Hence, 85% of the 1200 people interviewed enjoy either their jobs or their personal lives.

b) From the law of partitions,

$$P(A) = P(B \cap A) + P(B^c \cap A) = P(A \cap B) + P(A \cap B^c) = 0.04 + 0.80 = 0.84.$$

Hence, 84% of the 1200 people interviewed enjoy their jobs.
c) From part (b), the complementation rule, and the law of partitions,

$$\begin{aligned} 0.16 = 1 - 0.84 = 1 - P(A) = P(A^c) &= P\big(B \cap A^c\big) + P\big(B^c \cap A^c\big) \\ &= P\big(B \cap A^c\big) + P\big(A^c \cap B^c\big) = P\big(B \cap A^c\big) + 0.15. \end{aligned}$$

Hence, $P(B \cap A^c) = 0.16 - 0.15 = 0.01$. Therefore, 1% of the 1200 people interviewed enjoy their personal lives but not their jobs.
d) From the law of partitions and part (c),

$$P(B) = P(A \cap B) + P(A^c \cap B) = P(A \cap B) + P(B \cap A^c) = 0.04 + 0.01 = 0.05.$$

Hence, 5% of the 1200 people interviewed enjoy their personal lives.

2.69 We use the information and notation from Example 2.31.
a) Let A denote the event that the household selected gets either the *Times* or the *Herald*, but not both. We have $A = (T \cap H^c) \cup (T^c \cap H)$, and the two events in the union are mutually exclusive. From the law of partitions,

$$P\big(T \cap H^c\big) = P(T) - P(T \cap H) = 0.470 - 0.119 = 0.351$$

and

$$P\big(T^c \cap H\big) = P(H) - P(T \cap H) = 0.334 - 0.119 = 0.215.$$

Hence, from the additivity axiom,

$$P(A) = P\big((T \cap H^c) \cup (T^c \cap H)\big) = P\big(T \cap H^c\big) + P\big(T^c \cap H\big) = 0.351 + 0.215 = 0.566.$$

b) Let B denote the event that the household selected gets exactly one of the three newspapers. We have $B = (T \cap H^c \cap E^c) \cup (T^c \cap H \cap E^c) \cup (T^c \cap H^c \cap E)$, and the three events in the union are mutually exclusive. From the law of partitions,

$$\begin{aligned} P\big(T \cap H^c \cap E^c\big) &= P\big(T \cap H^c\big) - P\big(T \cap H^c \cap E\big), \\ P\big(T \cap H^c\big) &= P(T) - P(T \cap H), \end{aligned}$$

and

$$P\big(T \cap H^c \cap E\big) = P(T \cap E) - P(T \cap H \cap E).$$

Hence,

$$\begin{aligned} P\big(T \cap H^c \cap E^c\big) &= \big(P(T) - P(T \cap H)\big) - \big(P(T \cap E) - P(T \cap H \cap E)\big) \\ &= P(T) - P(T \cap H) - P(T \cap E) + P(T \cap H \cap E) \\ &= 0.470 - 0.119 - 0.151 + 0.048 \\ &= 0.248. \end{aligned}$$

Proceeding similarly, we find that $P(T^c \cap H \cap E^c) = 0.159$ and $P(T^c \cap H^c \cap E) = 0.139$. Therefore, by the additivity axiom,

$$\begin{aligned} P(B) &= P\big((T \cap H^c \cap E^c) \cup (T^c \cap H \cap E^c) \cup (T^c \cap H^c \cap E)\big) \\ &= P\big(T \cap H^c \cap E^c\big) + P\big(T^c \cap H \cap E^c\big) + P\big(T^c \cap H^c \cap E\big) \\ &= 0.248 + 0.159 + 0.139 \\ &= 0.546. \end{aligned}$$

c) Let C denote the event that the household selected gets none of the three newspapers. Then we have $C = T^c \cap H^c \cap E^c$. Applying De Morgan's law, the complementation rule, and the result of Example 2.31, we get

$$P(C) = P\big(T^c \cap H^c \cap E^c\big) = P\big((T \cup H \cup E)^c\big) = 1 - P(T \cup H \cup E) = 1 - 0.824 = 0.176.$$

d) Let D denote the event that the household selected gets the *Times* and *Herald*, but not the *Examiner*. Then $D = T \cap H \cap E^c$. Hence, from the law of partitions,

$$P(D) = P\big(T \cap H \cap E^c\big) = P(T \cap H) - P(T \cap H \cap E) = 0.119 - 0.048 = 0.071.$$

e) For $k = 0, 1, 2, 3$, let E_k denote the event that the household selected gets exactly k of the three newspapers. We note that the E_ks form a partition of the sample space. Using the certainty axiom, the additivity axiom, and the results of parts (b) and (c), we deduce that

$$\begin{aligned} 1 &= P(\Omega) = P(E_0 \cup E_1 \cup E_2 \cup E_3) = P(E_0) + P(E_1) + P(E_2) + P(E_3) \\ &= 0.176 + 0.546 + P(E_2) + 0.048. \end{aligned}$$

Hence, $P(E_2) = 1 - (0.176 + 0.546 + 0.048) = 0.230$.

2.70

a) Let A and B be events. Let us set $A_1 = A$ and $A_2 = B$. Applying the inclusion–exclusion principle with $N = 2$, we get

$$\begin{aligned} P(A \cup B) &= P\left(\bigcup_{n=1}^{2} A_n\right) = \sum_{k=1}^{2} P(A_k) - \sum_{k_1 < k_2} P(A_{k_1} \cap A_{k_2}) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ &= P(A) + P(B) - P(A \cap B), \end{aligned}$$

which is the general addition rule.

b) Let A and B be mutually exclusive events. Applying the general addition rule and Equation (2.10) on page 64, we get

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(\emptyset) = P(A) + P(B) - 0 \\ &= P(A) + P(B), \end{aligned}$$

which is the additivity axiom when applied to two (mutually exclusive) events.

c) Let $A_1, A_2, \ldots, A_N$ be mutually exclusive events. For $2 \le n \le N$ and $1 \le k_1 < k_2 < \cdots < k_n \le N$, we have, because the A_ns are mutually exclusive, that

$$A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n} \subset A_{k_1} \cap A_{k_2} = \emptyset.$$

Therefore, $A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n} = \emptyset$ and, hence, from Equation (2.10), we have

$$P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n}) = 0.$$

Applying now the inclusion–exclusion principle, we get

$$\begin{aligned} P\left(\bigcup_{n=1}^{N} A_n\right) &= \sum_{k=1}^{N} P(A_k) - \left(\sum_{k_1 < k_2} 0\right) + \cdots + (-1)^{n+1}\left(\sum_{k_1 < k_2 < \cdots < k_n} 0\right) + \cdots + (-1)^{N+1} \cdot 0 \\ &= \sum_{k=1}^{N} P(A_k), \end{aligned}$$

which is the additivity axiom when applied to N (mutually exclusive) events.

2.71 Let B denote the event that at least one student got the right quiz back. Then $B = \bigcup_{n=1}^{4} A_n$. Therefore, by the inclusion–exclusion principle,

$$\begin{aligned}
P(B) &= \sum_{k=1}^{4} P(A_k) - \sum_{k_1<k_2} P(A_{k_1} \cap A_{k_2}) + \sum_{k_1<k_2<k_3} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) - P(A_1 \cap A_2 \cap A_3 \cap A_4) \\
&= P(A_1) + P(A_2) + P(A_3) + P(A_4) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) \\
&\quad - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4) + P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) \\
&\quad + P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_2 \cap A_3 \cap A_4).
\end{aligned}$$

Theory Exercises

2.72 From Exercise 2.70(a), for $N = 2$, the inclusion–exclusion principle reduces to the general addition rule, which we proved on page 69. Assuming the truth of the inclusion–exclusion principle for N, we prove it for $N + 1$. From the general addition rule,

$$P\left(\bigcup_{n=1}^{N+1} A_n\right) = P\left(\left(\bigcup_{n=1}^{N} A_n\right) \cup A_{N+1}\right) = P\left(\bigcup_{n=1}^{N} A_n\right) + P(A_{N+1}) - P\left(\left(\bigcup_{n=1}^{N} A_n\right) \cap A_{N+1}\right),$$

or, in view of the distributive law for sets,

$$P\left(\bigcup_{n=1}^{N+1} A_n\right) = P\left(\bigcup_{n=1}^{N} A_n\right) + P(A_{N+1}) - P\left(\bigcup_{n=1}^{N} (A_n \cap A_{N+1})\right). \qquad (*)$$

Now, from the induction assumption,

$$\begin{aligned}
P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{k=1}^{N} P(A_k) &- \sum_{1\le k_1<k_2\le N} P(A_{k_1} \cap A_{k_2}) + \cdots \\
&+ (-1)^{n+1} \sum_{1\le k_1<k_2<\cdots<k_n\le N} P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n}) \\
&+ \cdots + (-1)^{N+1} P(A_1 \cap A_2 \cap \cdots \cap A_N)
\end{aligned}$$

and noting that, for $2 \le n \le N$ and $1 \le k_1 < k_2 < \cdots < k_n \le N$,

$$(A_{k_1} \cap A_{N+1}) \cap (A_{k_2} \cap A_{N+1}) \cap \cdots \cap (A_{k_n} \cap A_{N+1}) = A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n} \cap A_{N+1},$$

we also get

$$\begin{aligned}
P\left(\bigcup_{n=1}^{N} (A_n \cap A_{N+1})\right) = \sum_{k=1}^{N} P(A_k \cap A_{N+1}) &- \sum_{1\le k_1<k_2\le N} P(A_{k_1} \cap A_{k_2} \cap A_{N+1}) + \cdots \\
&+ (-1)^{n+1} \sum_{1\le k_1<k_2<\cdots<k_n\le N} P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n} \cap A_{N+1}) \\
&+ \cdots + (-1)^{N+1} P(A_1 \cap A_2 \cap \cdots \cap A_N \cap A_{N+1}).
\end{aligned}$$

Upon substituting the right sides of the preceding two equations for the corresponding left sides into Equation $(*)$, we see the truth of the inclusion-exclusion principle for $N + 1$.

2.73

a) As $B_1 = A_1$, we see that B_1 is the event that A_1 occurs. For $n \geq 2$, we have $B_n = A_n \cap A_{n-1}^c$, so that, in this case, B_n is the event that A_n occurs but A_{n-1} doesn't.

b) Let $m, n \in \mathcal{N}$ with $m \neq n$, say, $m < n$. Then, as $m \leq n-1$, we have $A_m \subset A_{n-1}$. Therefore,

$$B_m \cap B_n = \left(A_m \cap A_{m-1}^c\right) \cap \left(A_n \cap A_{n-1}^c\right) \subset A_m \cap A_{n-1}^c \subset A_{n-1} \cap A_{n-1}^c = \emptyset.$$

Thus, $B_m \cap B_n = \emptyset$, and we have established that $B_1, B_2, \dots$ are mutually exclusive.

Now, because $B_n \subset A_n$ for all $n \in \mathcal{N}$, we clearly have $\bigcup_{n=1}^\infty B_n \subset \bigcup_{n=1}^\infty A_n$. Conversely, suppose that $\omega \in \bigcup_{n=1}^\infty A_n$. Then there is an $n \in \mathcal{N}$ such that $\omega \in A_n$. Let m be the smallest such n. Then $\omega \in A_m \cap A_{m-1}^c = B_m$ and, so, $\omega \in \bigcup_{n=1}^\infty B_n$. Therefore, $\bigcup_{n=1}^\infty B_n \supset \bigcup_{n=1}^\infty A_n$. Hence, we have shown that $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n$.

c) From the law of partitions and the fact that $A_{n-1} \subset A_n$, we deduce that, for $n \geq 2$,

$$P(B_n) = P(A_n \cap A_{n-1}^c) = P(A_n) - P(A_n \cap A_{n-1}) = P(A_n) - P(A_{n-1}).$$

d) From part (b), the additivity axiom, and part (c),

$$\begin{aligned} P\left(\bigcup_{n=1}^\infty B_n\right) &= \sum_{n=1}^\infty P(B_n) = P(A_1) + \sum_{n=2}^\infty \big(P(A_n) - P(A_{n-1})\big) \\ &= P(A_1) + \lim_{n\to\infty} \sum_{k=2}^n \big(P(A_k) - P(A_{k-1})\big) = P(A_1) + \lim_{n\to\infty} \big(P(A_n) - P(A_1)\big) \\ &= \lim_{n\to\infty} P(A_n). \end{aligned}$$

e) From parts (b) and (d),

$$P\left(\bigcup_{n=1}^\infty A_n\right) = P\left(\bigcup_{n=1}^\infty B_n\right) = \lim_{n\to\infty} P(A_n).$$

Hence, Proposition 2.11(a) holds.

f) Let $A_1, A_2, \dots$ be events such that $A_1 \supset A_2 \supset \cdots$. Setting $C_n = A_n^c$ for all $n \in \mathcal{N}$, we have that $C_1 \subset C_2 \subset \cdots$. Consequently, from De Morgan's law, the complementation rule, and Proposition 2.11(a), we get

$$\begin{aligned} P\left(\bigcap_{n=1}^\infty A_n\right) &= 1 - P\left(\left(\bigcap_{n=1}^\infty A_n\right)^c\right) = 1 - P\left(\bigcup_{n=1}^\infty A_n^c\right) = 1 - P\left(\bigcup_{n=1}^\infty C_n\right) = 1 - \lim_{n\to\infty} P(C_n) \\ &= 1 - \lim_{n\to\infty} P(A_n^c) = \lim_{n\to\infty} \big(1 - P(A_n^c)\big) = \lim_{n\to\infty} P(A_n). \end{aligned}$$

Hence, Proposition 2.11(b) holds.

2.74

a) From the certainty axiom, domination principle, and general addition rule, we deduce that

$$1 = P(\Omega) \geq P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

Therefore, $P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$, which is Bonferroni's inequality for $N = 2$.

b) From part (a), Bonferroni's inequality holds for $N = 2$. Assuming its truth for N, we prove it for $N + 1$. We have

$$P\left(\bigcap_{n=1}^{N+1} A_n\right) = P\left(\left(\bigcap_{n=1}^{N} A_n\right) \cap A_{N+1}\right) \geq P\left(\bigcap_{n=1}^{N} A_n\right) + P(A_{N+1}) - 1$$

$$\geq \sum_{n=1}^{N} P(A_n) - (N-1) + P(A_{N+1}) - 1 = \sum_{n=1}^{N+1} P(A_n) - \big((N+1) - 1\big),$$

as required.

2.75

a) From the general addition rule and nonnegativity axiom,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2).$$

b) From part (a), Boole's inequality holds for $N = 2$. Assuming its truth for N, we prove it for $N + 1$. We have

$$P\left(\bigcup_{n=1}^{N+1} A_n\right) = P\left(\left(\bigcup_{n=1}^{N} A_n\right) \cup A_{N+1}\right) \leq P\left(\bigcup_{n=1}^{N} A_n\right) + P(A_{N+1})$$

$$\leq \sum_{n=1}^{N} P(A_n) + P(A_{N+1}) = \sum_{n=1}^{N+1} P(A_n),$$

as required.

c) Let $B_n = \bigcup_{k=1}^{n} A_k$ for each $n \in \mathcal{N}$. We note that $A_n \subset B_n$ for all $n \in \mathcal{N}$ and that $B_1 \subset B_2 \subset \cdots$. Now, as $A_n \subset B_n$ for each $n \in \mathcal{N}$, we clearly have $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$. Conversely, let us suppose that $\omega \in \bigcup_{n=1}^{\infty} B_n$. Then there is an $m \in \mathcal{N}$ such that $\omega \in B_m$, which, in turn, implies that $\omega \in A_k$ for some $1 \leq k \leq m$. Therefore, $\omega \in \bigcup_{n=1}^{\infty} A_n$ and, hence, $\bigcup_{n=1}^{\infty} A_n \supset \bigcup_{n=1}^{\infty} B_n$. We have thus shown that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Applying now Proposition 2.11(a) and part (b), we deduce that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} A_k\right) \leq \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) = \sum_{n=1}^{\infty} P(A_n).$$

d) Applying the complementation rule, De Morgan's law, Boole's inequality, and the complementation rule again, we get

$$P\left(\bigcap_{n=1}^{N} A_n\right) = 1 - P\left(\left(\bigcap_{n=1}^{N} A_n\right)^c\right) = 1 - P\left(\bigcup_{n=1}^{N} A_n^c\right)$$

$$\geq 1 - \sum_{n=1}^{N} P(A_n^c) = 1 - \sum_{n=1}^{N}\big(1 - P(A_n)\big) = 1 - N + \sum_{n=1}^{N} P(A_n)$$

$$= \sum_{n=1}^{N} P(A_n) - (N-1).$$

Advanced Exercises

2.76

a) For each $n \in \mathcal{N}$, let A_n denote the event that the first head occurs on the nth toss. When a balanced coin is tossed n times, 2^n equally likely outcomes are possible. Hence,

$$P(A_n) = P(\{\underbrace{\text{TTT}\ldots\text{T}}_{n-1 \text{ times}}\text{H}\}) = \frac{1}{2^n}.$$

The events $A_1, A_2, \ldots$ are mutually exclusive and, clearly, $A = \bigcup_{n=1}^{\infty} A_n$. Therefore, by the additivity axiom and the formula for a geometric series,

$$P(A) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1 - 1/2} = 1.$$

b) Let O denote the event that the first head occurs on an odd-numbered toss. Then $O = \bigcup_{n=1}^{\infty} A_{2n-1}$ and, hence, by the additivity axiom and the formula for a geometric series,

$$P(O) = \sum_{n=1}^{\infty} P(A_{n-1}) = \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} = 2\sum_{n=1}^{\infty} \frac{1}{4^n} = 2 \cdot \frac{1/4}{1 - 1/4} = \frac{2}{3}.$$

c) Let E denote the event that the first head occurs on an even-numbered toss. We note that $A = E \cup O$ and that E and O are mutually exclusive. Consequently, by the additivity axiom and parts (a) and (b), we have

$$P(E) = P(A) - P(O) = 1 - \frac{2}{3} = \frac{1}{3}.$$

2.77 From Exercise 2.18, we know that $A^* = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$. For each $n \in \mathcal{N}$, set $B_n = \bigcup_{k=n}^{\infty} A_k$. Then $B_1 \supset B_2 \supset \cdots$. Hence, from the nonnegativity axiom, Proposition 2.11(b), Boole's inequality, and the fact that the tail of a convergent series converges to 0, we get that

$$0 \le P(A^*) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) = 0.$$

Therefore, $P(A^*) = 0$.

2.78

a) We prove, using mathematical induction, that

$$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n).$$

By finite additivity, the result holds for $N = 2$. Assuming its truth for N, we prove it for $N + 1$. So, assume that $A_1, \ldots, A_{N+1}$ are mutually exclusive events. Then $A_1, \ldots, A_N$ are mutually exclusive.

Furthermore, from the distributive law for sets,

$$\left(\bigcup_{n=1}^{N} A_n\right) \cap A_{N+1} = \bigcup_{n=1}^{N}(A_n \cap A_{N+1}) = \bigcup_{n=1}^{N} \emptyset = \emptyset,$$

so that $\bigcup_{n=1}^{N} A_n$ and A_{N+1} are mutually exclusive. Therefore, from finite additivity and the induction assumption,

$$P\left(\bigcup_{n=1}^{N+1} A_n\right) = P\left(\left(\bigcup_{n=1}^{N} A_n\right) \cup A_{N+1}\right) = P\left(\bigcup_{n=1}^{N} A_n\right) + P(A_{N+1})$$
$$= \sum_{n=1}^{N} P(A_n) + P(A_{N+1}) = \sum_{n=1}^{N+1} P(A_n),$$

as required.

b) In view of part (a), we need only establish countable additivity in the case where we have an infinite sequence, $A_1, A_2, \ldots,$ of mutually exclusive events. For each $n \in \mathcal{N}$, let $B_n = \bigcup_{k=1}^{n} A_k$. Then $B_1 \subset B_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Applying now the assumed continuity and finite additivity, we conclude that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} A_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) = \sum_{n=1}^{\infty} P(A_n).$$

c) From part (b), we know that finite additivity and continuity together imply countable additivity. Conversely, countable additivity clearly implies finite additivity and, in view of Proposition 2.11, it also implies continuity.

Review Exercises for Chapter 2

Basic Exercises

2.79 In the ordered-pair representation, the black die comes up 2 if and only if the first member of the ordered pair is 2. Hence, the event that the black die comes up 2 is

$$\{\,(2, y) : y \in \{1, 2, 3, 4, 5, 6\}\,\} = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)\}.$$

2.80

a) The sample space for this random experiment consists of all subsets of size 2 of the six club members, where the elements of such a subset represent the two members selected as co-chairs. Hence,

$$\Omega = \big\{\{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,f\}, \{b,c\}, \{b,d\}, \{b,e\}, \{b,f\}, \{c,d\}, \{c,e\}, \{c,f\}, \{d,e\}, \{d,f\}, \{e,f\}\big\}.$$

b) Let A denote the event that at least one of a and f is chosen as a co-chair. As a subset of the sample space, A consists of all elements of Ω that contain either a or f (or both). Hence,

$$A = \big\{\{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,f\}, \{b,f\}, \{c,f\}, \{d,f\}, \{e,f\}\big\}.$$

c) The sample space for this random experiment consists of all subsets of size 2 of the six club members in which one is a man and the other is a woman. The elements of such a subset represent the two members selected as co-chairs. Hence, in this case,

$$\Omega = \{\{a, d\}, \{a, e\}, \{a, f\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, d\}, \{c, e\}, \{c, f\}\}.$$

Let A denote the event that at least one of a and f is chosen as a co-chair. As a subset of the sample space, A consists of all elements of Ω that contain either a or f (or both). Hence,

$$A = \{\{a, d\}, \{a, e\}, \{a, f\}, \{b, f\}, \{c, f\}\}.$$

d) The sample space for this random experiment consists of all ordered pairs from the six club members, where the first and second entries of such an ordered pair represent the chair and vice-chair selected, respectively. Hence,

$$\begin{aligned}\Omega = \{&(a, b), (a, c), (a, d), (a, e), (a, f), (b, a), (b, c), (b, d), (b, e), (b, f),\\ &(c, a), (c, b), (c, d), (c, e), (c, f), (d, a), (d, b), (d, c), (d, e), (d, f),\\ &(e, a), (e, b), (e, c), (e, d), (e, f), (f, a), (f, b), (f, c), (f, d), (f, e)\}.\end{aligned}$$

Let A denote the event that at least one of a and f is chosen as one of the two officers. As a subset of the sample space, A consists of all elements of Ω that contain either a or f (or both). Hence,

$$\begin{aligned}A = \{&(a, b), (a, c), (a, d), (a, e), (a, f), (b, a), (b, f), (c, a), (c, f), (d, a),\\ &(d, f), (e, a), (e, f), (f, a), (f, b), (f, c), (f, d), (f, e)\}.\end{aligned}$$

2.81 There are six possible categories in the cross classification, which we list in the following table:

Category, k	Description
1	foreign-made compact
2	foreign-made midsize
3	foreign-made fullsize
4	U.S.-made compact
5	U.S.-made midsize
6	U.S.-made fullsize

Now, let x_k denote the number of the 100 cars you observe that are of category k. Each possible outcome of the random experiment can then be represented as an ordered six-tuple, $(x_1, x_2, x_3, x_4, x_5, x_6)$. Hence, a sample space for this random experiment is

$$\Omega = \left\{ (x_1, x_2, x_3, x_4, x_5, x_6) : x_k \in \{0, 1, 2, \ldots\} \text{ for } 1 \le k \le 4, \text{ and } \textstyle\sum_{k=1}^{6} x_k = 100 \right\}.$$

2.82

a) The events that at least five men are chosen and at least five women are chosen are not mutually exclusive. Indeed, both events occur if five, six, or seven men are chosen.

b) The events that at least five men are chosen and at least eight women are chosen are mutually exclusive. Indeed, if at least five men are chosen, then at most seven women can be chosen.

c) The events that five men and seven women are chosen, four men and eight women are chosen, and three men and nine women are chosen are mutually exclusive. Indeed each of these three events specifies a different number of men (and women) that are chosen and, hence, no two of the events can occur simultaneously.

d) The events that the first person on the list of 20 is among those chosen, the second person on the list of 20 is among those chosen, and the third person on the list of 20 is among those chosen are not mutually exclusive. In fact, all three events can occur simultaneously, namely, if the first three people on the list are among the 12 people chosen for the jury.

2.83

a) A typical outcome of the random experiment of assigning the three alternates can be represented as an ordered triple, (x, y, z), where x, y, and z denote the first, second, and third alternates, respectively. Hence, a sample space is

$$\begin{aligned}\Omega &= \Big\{ (x, y, z) : x, y, z \in \{\text{A, B, C}\} \text{ and } x \neq y \neq z \Big\} \\ &= \{(\text{A, B, C}), (\text{A, C, B}), (\text{B, A, C}), (\text{B, C, A}), (\text{C, A, B}), (\text{C, B, A})\}.\end{aligned}$$

b) Person A is the first alternate if and only if the first entry of the ordered triple is A. Hence, as a subset of the sample space, the event that A is the first alternate is {(A, B, C), (A, C, B)}.
c) Event $A^c \cap B^c \cap C^c$ is that A is not the first alternate, B is not the second alternate, and C is not the third alternate. As a subset of the sample space, we have $A^c \cap B^c \cap C^c = \{(\text{B, C, A}), (\text{C, A, B})\}$.
d) Event $A \cap B \cap C$ is that A is the first alternate, B is the second alternate, and C is the third alternate. As a subset of the sample space, we have $A \cap B \cap C = \{(\text{A, B, C})\}$.
e) Event $A \cap C$ is that A is the first alternate and C is the third alternate. As a subset of the sample space, we have $A \cap C = \{(\text{A, B, C})\}$. We note that $A \cap B \cap C = A \cap C$.
f) Because who among A, B, and C will be the first alternate, who will be the second, and who will be the third will be decided by chance, an equal-likelihood model is appropriate here. Hence, each of the six possible outcomes should be assigned the same probability, namely, probability 1/6.
g) As an equal-likelihood model is appropriate, we have, for each event E, that

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

Referring now to parts (c)–(e), we get

$$\begin{aligned}P(A^c \cap B^c \cap C^c) &= \frac{N(A^c \cap B^c \cap C^c)}{6} = \frac{N\big(\{(\text{B, C, A}), (\text{C, A, B})\}\big)}{6} = \frac{2}{6} = \frac{1}{3}, \\ P(A \cap B \cap C) &= \frac{N(A \cap B \cap C)}{6} = \frac{N\big(\{(\text{A, B, C})\}\big)}{6} = \frac{1}{6}, \\ P(A \cap C) &= \frac{N(A \cap C)}{6} = \frac{N\big(\{(\text{A, B, C})\}\big)}{6} = \frac{1}{6}.\end{aligned}$$

2.84 Answers will vary. Following is what we did and got. In each of parts (a)–(c), we used a random-number generator to obtain 1000 random decimal digits; in other words, in each part, we simulated 1000 times the random experiment of selecting a number at random from the set $\{0, 1, 2, \ldots, 9\}$.
a) In this simulation, event A (0, 1, 2, or 3) occurred 395 times and, hence, the proportion of times that event A occurred is 0.395. In view of the frequentist interpretation of probability, this proportion estimates $P(A)$.
b) In this simulation, event B (7 or 8) occurred 191 times and, hence, the proportion of times that event B occurred is 0.191. In view of the frequentist interpretation of probability, this proportion estimates $P(B)$.
c) In this simulation, event $A \cup B$ (0, 1, 2, 3, 7, or 8) occurred 608 times and, hence, the proportion of times that event $A \cup B$ occurred is 0.608. In view of the frequentist interpretation of probability, this proportion estimates $P(A \cup B)$.
d) From the additivity axiom, $P(A \cup B) = P(A) + P(B)$ and, from our estimates in parts (a)–(c), $P(A \cup B) \approx 0.608$ and $P(A) + P(B) \approx 0.395 + 0.191 = 0.586$. Hence, for these simulations, the results are consistent with the additivity axiom provided that we round to one decimal place. Given that the number of simulations is only moderate, the discrepancy here is not surprising.

e) Repeating parts (a)–(c), but with 10,000 simulations each time, we obtained $P(A \cup B) \approx 0.5961$ and $P(A) + P(B) \approx 0.3977 + 0.2009 = 0.5986$. Hence, for these simulations, the results are consistent with the additivity axiom provided that we round to two decimal places.

2.85 The completed table is as follows:

Event	Description
$A \cap B^c$	A occurs but B doesn't.
$A \cup B^c$	Either A occurs or B doesn't.
$A^c \cap B^c$	Neither A nor B occurs.
$(A \cap B^c) \cup (A^c \cap B)$	Exactly one of A and B occurs.
$A \cap B \cap C$	Events A, B, and C occur.
$A \cap (B \cup C)$	A occurs and either B or C occurs.
$A \cup (B \cap C)$	Either A occurs or both B and C occur.
$A \cup B \cup C$	At least one of A, B, and C occurs.
$A^c \cup B^c \cup C^c$	At least one of A, B, and C does not occur.
$\bigcap_n A_n$	All of $A_1, A_2, \ldots$ occur.
$\bigcup_n A_n$	At least one of $A_1, A_2, \ldots$ occurs.

2.86 Exactly one of event A and event B occurs if and only if either event A occurs and event B doesn't or event B occurs and event A doesn't, which is the case if and only if either both event A and event B^c occur or both event B and event A^c occur, that is, if and only if event $(A \cap B^c) \cup (B \cap A^c)$ occurs. Other ways to express the event in question are possible. For instance, that event occurs if and only if at least one but not both of event A and event B occur, which is the event $(A \cup B) \cap (A^c \cup B^c)$.

2.87

a) True. If A, B, and C are (pairwise) mutually exclusive, then, in particular, we have $A \cap B = \emptyset$, which means that A and B are mutually exclusive.

b) False. It's possible for A and B to be mutually exclusive without A, B, and C being (pairwise) mutually exclusive. For instance, any three events, A, B, and C, such that $A \cap B = \emptyset$ and $A \cap C \neq \emptyset$ will provide such a situation.

2.88

a) Because $0 \le p \le 1$, the numbers p^2, $p(1-p)$, $p(1-p)$, and $(1-p)^2$ are nonnegative. They also sum to 1 because

$$\begin{aligned} 1 = 1^2 &= \big(p + (1-p)\big)^2 = p^2 + 2p(1-p) + (1-p)^2 \\ &= p^2 + p(1-p) + p(1-p) + (1-p)^2. \end{aligned}$$

Therefore, from Proposition 2.3 on page 43, the specified assignment determines a unique probability measure on Ω.

b) Referring to Proposition 2.2 on page 42, we see, for instance, that

$$\begin{aligned} P(\{\text{HH, HT, TH}\}) &= P(\{\text{HH}\}) + P(\{\text{HT}\}) + P(\{\text{TH}\}) = p^2 + p(1-p) + p(1-p) \\ &= p\big(p + (1-p) + (1-p)\big) = p(2-p). \end{aligned}$$

Proceeding similarly, we obtain the following table:

Event	Probability	Event	Probability
$\emptyset$	0	{HT, TH}	$2p(1-p)$
{HH}	p^2	{HT, TT}	$1-p$
{HT}	$p(1-p)$	{TH, TT}	$1-p$
{TH}	$p(1-p)$	{HH, HT, TH}	$p(2-p)$
{TT}	$(1-p)^2$	{HH, HT, TT}	$1-p+p^2$
{HH, HT}	p	{HH, TH, TT}	$1-p+p^2$
{HH, TH}	p	{HT, TH, TT}	$1-p^2$
{HH, TT}	$1-2p+2p^2$	Ω	1

2.89
a) We can represent a typical outcome of the random experiment as a triple (x_1, x_2, x_3), where each x_k is either w or b depending on whether component k is working or broken, respectively. Hence, a sample space is

$$\begin{aligned}\Omega &= \left\{\, (x_1, x_2, x_3) : x_k \in \{w, b\} \text{ for } k = 1, 2, 3 \,\right\} \\ &= \{(w,w,w), (w,w,b), (w,b,w), (w,b,b), (b,w,w), (b,w,b), (b,b,w), (b,b,b)\}.\end{aligned}$$

As each component is equally likely to be working or broken, each of the eight possible outcomes is equally likely. Thus, a classical probability model is appropriate here and we have, for each event E, that

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{8}.$$

b) Let A denote the event that all three components are broken. Then $A = \{(b,b,b)\}$ and

$$P(A) = P\big(\{(b,b,b)\}\big) = \frac{N\big(\{(b,b,b)\}\big)}{8} = \frac{1}{8}.$$

c) Let B denote the event that the device is functioning, that is, that at least two of the three components are working. Then $B = \{(w,w,w), (w,w,b), (w,b,w), (b,w,w)\}$ and

$$\begin{aligned}P(B) &= P\big(\{(w,w,w), (w,w,b), (w,b,w), (b,w,w)\}\big) \\ &= \frac{N\big(\{(w,w,w), (w,w,b), (w,b,w), (b,w,w)\}\big)}{8} = \frac{4}{8} = \frac{1}{2}.\end{aligned}$$

2.90 The sample space, Ω, is the unit disk and, because a point is being chosen at random, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\pi},$$

where $|E|$ denotes the area of the set E.
a) Let A denote the event that the distance from the point chosen to the center of the disk exceeds 1/2. Then event A^c is that the distance from the point chosen to the center of the disk is at most 1/2. Thus, A^c is the disk of radius $1/2$ centered at the origin, which has area $\pi \cdot (1/2)^2 = \pi/4$. Consequently, from the complementation rule,

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{\pi} = 1 - \frac{\pi/4}{\pi} = \frac{3}{4}.$$

b) Let B denote the event that the sum of the magnitudes of the two coordinates of the point chosen exceeds 1/2. Then event B^c is that the sum of the magnitudes of the two coordinates of the point chosen is at most 1/2. Thus, B^c is the square with vertices $(1/2, 0)$, $(0, 1/2)$, $(-1/2, 0)$, and $(0, -1/2)$, which has area $\left(1/\sqrt{2}\right)^2 = 1/2$. Consequently, from the complementation rule,

$$P(B) = 1 - P(B^c) = 1 - \frac{|B^c|}{\pi} = 1 - \frac{1/2}{\pi} = 1 - \frac{1}{2\pi} \approx 0.841.$$

c) Let C denote the event that the maximum of the magnitudes of the two coordinates of the point chosen exceeds 1/2. Then event C^c is that the maximum of the magnitudes of the two coordinates of the point chosen is at most 1/2. Thus, C^c is the square with vertices $(1/2, 1/2)$, $(-1/2, 1/2)$, $(-1/2, -1/2)$, and $(1/2, -1/2)$, which has area $1^2 = 1$. Consequently, from the complementation rule,

$$P(C) = 1 - P(C^c) = 1 - \frac{|C^c|}{\pi} = 1 - \frac{1}{\pi} \approx 0.682.$$

2.91

a) We have $1/2^n \geq 0$ for all $n \in \mathcal{N}$ and, moreover, from the formula for a geometric series,

$$\sum_{n\in\mathcal{N}} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1/2}{1 - 1/2} = 1.$$

Hence, from Proposition 2.3 on page 43, the specified probability assignment is legitimate.

b) Let O denote the event that the outcome is an odd number. Then $O = \{1, 3, 5, \dots\}$ and, hence, from Proposition 2.2 on page 42 and the formula for a geometric series,

$$P(O) = \sum_{\omega\in O} P(\{\omega\}) = \sum_{n=0}^{\infty} P(\{2n+1\}) = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{2} \cdot \frac{1}{1 - 1/4} = \frac{2}{3}.$$

2.92

a) If event B_6 occurs, then, after the sixth toss, you have still not reached your goal, which implies that, after the fifth toss, you have still not reached your goal, that is, that event B_5 occurs. Hence, $B_6 \subset B_5$ and, therefore, from the domination principle, we have $P(B_6) \leq P(B_5)$.

b) Let A denote the event that you never reach your goal. We note that event A occurs if and only if after the nth toss, you have still not reached your goal for each $n \geq 3$, that is, if and only if event B_n occurs for all $n \geq 3$. Thus, $A = \bigcap_{n=3}^{\infty} B_n$. Arguing as in part (a), we see that $B_3 \supset B_4 \supset \cdots$. Therefore, from the continuity property of a probability measure, specifically, Proposition 2.11(b) on page 74, we have

$$P(A) = P\left(\bigcap_{n=3}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n).$$

c) Let A denote the event that you never reach your goal. We note that, if either the first three tosses are heads or the first toss is a tail followed by five consecutive heads, then you reach your goal, that is, event A^c occurs. The probability that the first three tosses are heads is $(1/2)^3 = 1/8$ and the probability that the first toss is a tail followed by five consecutive heads is $(1/2)^6 = 1/64$. Hence, from the complementation rule and the domination principle,

$$P(A) = 1 - P(A^c) \leq 1 - \left(\frac{1}{8} + \frac{1}{64}\right) < 1 - \frac{1}{8} = \frac{7}{8}.$$

2.93 Let us measure time in hours after 3:00 P.M. A typical outcome of the random experiment can be represented as an ordered pair (x, y), where x and y represent the arrival times of woman 1 and woman 2, respectively. Thus, a sample space is $\Omega = [0, 2] \times [0, 2]$. Moreover, because each arrival time is equally likely between 3:00 P.M. and 5:00 P.M. and the time of arrival of one woman doesn't affect that of the other, a geometric probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{2 \cdot 2} = \frac{|E|}{4},$$

where $|E|$ is the area of the set E. Now, let A denote the event that the two women meet. Then we have $A = \{ (x, y) \in \Omega : |x - y| \leq 2/3$. We see that A^c consists of (the interior of) two triangles, one with vertices $(0, 2/3)$, $(0, 2)$, and $(4/3, 2)$, and the other with vertices $(2/3, 0)$, $(2, 0)$, and $(2, 4/3)$. Hence, we have $|A^c| = 2 \cdot (4/3)^2/2 = 16/9$. Consequently, by the complementation rule,

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{4} = 1 - \frac{16/9}{4} = \frac{5}{9}.$$

2.94 Estimating the probability of rain (and, more generally, weather prediction) is a complex issue. Answers will vary among all three methods for specifying probabilities—probability models, empirical probability, and subjective probability—and combinations thereof.

2.95
a) Empirical probability. Specifically, we would observe a large number of bolts produced by the manufacturing process and use the proportion of those bolts that are defective as (our estimate of) the probability of a defective bolt.
b) Answers will vary. Some might say that subjective probability, based on an educated guess or intuition, is the only way to assign the probability of a specified horse finishing first in a particular horse race. Others might argue that empirical probability, based, say, on information from the racing form, can be used. Still others might claim that a probability model, employing the win odds (obtained from the totalizator board) can be used.
c) Empirical probability. Specifically, we would observe a large number of horse races and use the proportion of those races in which the favorite wins as (our estimate of) the probability that the favorite in a horse race will finish first.
d) Answers will vary. See the solution to Exercise 2.94.

2.96 For a randomly selected household in this town, let A and H denote the events that the household chosen owns an automobile and owns a home, respectively. We know that $P(A) = 0.80$, $P(H) = 0.45$, and $P(A \cap H) = 0.35$.
a) The event that the household chosen owns either an automobile or a home but not both can be expressed as the union of two mutually exclusive events as follows: $(A \cap H^c) \cup (H \cap A^c)$. From the law of partitions,

$$P(A \cap H^c) = P(A) - P(A \cap H) = 0.80 - 0.35 = 0.45$$

and

$$P(H \cap A^c) = P(H) - P(H \cap A) = 0.45 - 0.35 = 0.10.$$

Therefore, from the additivity axiom,

$$P\big((A \cap H^c) \cup (H \cap A^c)\big) = P(A \cap H^c) + P(H \cap A^c) = 0.45 + 0.10 = 0.55.$$

b) The event that the household chosen owns neither an automobile nor a home can be expressed as $A^c \cap H^c$. From the general addition rule,

$$P(A \cup H) = P(A) + P(H) - P(A \cap H) = 0.80 + 0.45 - 0.35 = 0.90.$$

Therefore, from De Morgan's law and the complementation rule,

$$P\big(A^c \cap H^c\big) = P\big((A \cup H)^c\big) = 1 - P(A \cup H) = 1 - 0.90 = 0.10.$$

2.97 For the table presented in the problem statement, we sum each row, each column, and all the cell entries to obtain the following table:

Vehicle type	Country: U.S. C_1	Canada C_2	Mexico C_3	Total
Automobiles V_1	129,728	13,138	8,607	151,473
Motorcycles V_2	3,871	320	270	4,461
Trucks V_3	75,940	6,933	4,287	87,160
Total	209,539	20,391	13,164	243,094

In what follows, all frequencies are in thousands.

a) The number of vehicles that are not automobiles is $4{,}461 + 87{,}160 = 91{,}621$, obtained by summing the total number of motorcycles and trucks.

b) From the second-column total, we see that the number of Canadian vehicles is 20,391.

c) From the second-row total, we see that the number of motorcyles is 4461.

d) The cell in the second row and second column shows that the number of Canadian motorcycles is 320.

e) Summing the second-column total and second-row total and then subtracting the number of Canadian motorcyles (which is counted twice in summing the two forementioned totals), we find that the number of vehicles that are either Canadian or motorcycles is $20{,}391 + 4{,}461 - 320 = 24{,}532$.

f) C_1 is the event that the vehicle selected is in the United States; V_3 is the event that the vehicle selected is a truck; and $C_1 \cap V_3$ is the event that the vehicle selected is a United States truck.

Note: Because a North American vehicle is being selected at random, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{243{,}094}.$$

g) We have

$$P(C_1) = \frac{N(C_1)}{243{,}094} = \frac{209{,}539}{243{,}094} \approx 0.862,$$

$$P(V_3) = \frac{N(V_3)}{243{,}094} = \frac{87{,}160}{243{,}094} \approx 0.359,$$

$$P(C_1 \cap V_3) = \frac{N(C_1 \cap V_3)}{243{,}094} = \frac{75{,}940}{243{,}094} \approx 0.312.$$

h) From the table presented at the beginning of the solution to this exercise, we find that the number of vehicles that are either in the United States or are trucks is $209{,}539 + 87{,}160 - 75{,}940 = 220{,}759$. Hence,

$$P(C_1 \cup V_3) = \frac{N(C_1 \cup V_3)}{243{,}094} = \frac{220{,}759}{243{,}094} \approx 0.908.$$

i) Applying the general addition rule to the results of part (g), we get

$$P(C_1 \cup V_3) = P(C_1) + P(V_3) - P(C_1 \cap V_3) \approx 0.862 + 0.359 - 0.312 = 0.909.$$

The discrepancy between the answer here and that in part (h) is due to roundoff error.

j) Dividing each entry of the table presented at the beginning of the solution to this exercise by the grand total of 243,094, we obtain the following joint probability distribution:

Country

Vehicle type	U.S. C_1	Canada C_2	Mexico C_3	$P(V_i)$
Automobiles V_1	0.534	0.054	0.035	0.623
Motorcycles V_2	0.016	0.001	0.001	0.018
Trucks V_3	0.312	0.029	0.018	0.359
$P(C_j)$	0.862	0.084	0.054	1.000

2.98 A typical outcome of this random experiment can be represented as an ordered triple, (x_1, x_2, x_3), where x_k denotes the number of the ball obtained on draw k. Hence, a sample space is

$$\begin{aligned}\Omega &= \{\, (x_1, x_2, x_3) : x_k \in \{1, 2, 3\} \text{ for } k = 1, 2, 3, \text{ and } x_1 \neq x_2 \neq x_3 \,\} \\ &= \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}.\end{aligned}$$

Because the balls are selected at random, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

a) We have

$$\begin{aligned}P(A_1) &= P\big(\{(1,2,3), (1,3,2)\}\big) = \frac{N\big(\{(1,2,3), (1,3,2)\}\big)}{6} = \frac{2}{6} = \frac{1}{3}, \\ P(A_2) &= P\big(\{(1,2,3), (3,2,1)\}\big) = \frac{N\big(\{(1,2,3), (3,2,1)\}\big)}{6} = \frac{2}{6} = \frac{1}{3}, \\ P(A_3) &= P\big(\{(1,2,3), (2,1,3)\}\big) = \frac{N\big(\{(1,2,3), (2,1,3)\}\big)}{6} = \frac{2}{6} = \frac{1}{3}.\end{aligned}$$

Note: A much simpler way to obtain these probabilities is to use a *symmetry argument.* Specifically, as the balls are selected at random, the ith ball chosen is equally likely to be any one of the three balls; hence, the probability is $1/3$ that it will be ball i. Thus, we have $P(A_i) = 1/3$ for $i = 1, 2$, and 3.

b) Events A_1, A_2, and A_3 are not mutually exclusive. In fact, the outcome $(1, 2, 3)$ is common to all three of those events.

2.99 A sample space for this random experiment is

$$\Omega = \{\text{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,}\\ \text{THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}\}.$$

Because the coin is balanced, each of the 16 possible outcomes are equally likely. Hence, a classical probability model is appropriate here and we have, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{16}.$$

a) The event that the first tail is followed by two consecutive heads is {HHHH, HTHH, THHH, THHT} and, hence, the probability of that event is $4/16 = 1/4$. Note that the outcome HHHH vacuously satisfies the condition that the first tail is followed by two consecutive heads.
b) The event that a run of three or more heads occurs is {HHHH, HHHT, THHH} and, hence, the probability of that event is $3/16$.

2.100 A typical outcome of this random experiment can be represented as an ordered triple, (x_1, x_2, x_3), where x_k denotes the number of the husband with which wife k dances. Hence, a sample space is

$$\begin{aligned}\Omega &= \left\{\, (x_1, x_2, x_3) : x_k \in \{1, 2, 3\} \text{ for } k = 1, 2, 3, \text{ and } x_1 \neq x_2 \neq x_3 \,\right\}\\ &= \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}.\end{aligned}$$

Because the couples are paired at random, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

a) The event that each wife dances with her own husband is $\{(1, 2, 3)\}$ and, hence, the probability of that event is $1/6$.
b) The event that no wife dances with her own husband is $\{(2, 3, 1), (3, 1, 2)\}$ and, hence, the probability of that event is $2/6 = 1/3$.
c) The event that at least one wife dances with her own husband is the complement of the event that no wife dances with her own husband. Hence, from the complementation rule and part (b), the required probability is $1 - 1/3 = 2/3$.

2.101 Let W, D, and I denote the events that the customer will purchase a washer, dryer, and iron, respectively. We know that

$$\begin{gathered}P(W) = 0.4, \qquad P(D) = 0.3, \qquad P(I) = 0.23,\\ P(W \cap D) = 0.15, \qquad P(W \cap I) = 0.13, \qquad P(D \cap I) = 0.09,\\ P(W \cap D \cap I) = 0.05.\end{gathered}$$

a) Let A denote the event that the customer purchases none of the three items. Then $A = W^c \cap D^c \cap I^c$. From the inclusion–exclusion principle,

$$\begin{aligned}P(W \cup D \cup I) &= P(W) + P(D) + P(I) - P(W \cap D) - P(W \cap I) - P(D \cap I) + P(W \cap D \cap I)\\ &= 0.4 + 0.3 + 0.23 - 0.15 - 0.13 - 0.09 + 0.05 = 0.61.\end{aligned}$$

Hence, from De Morgan's law and the complementation rule, we have

$$P(A) = P\big(W^c \cap D^c \cap I^c\big) = P\big((W \cup D \cup I)^c\big) = 1 - P(W \cup D \cup I) = 1 - 0.61 = 0.39.$$

b) Let B denote the event that the customer purchases two or more items. We note that event B occurs if and only if the customer purchases both a washer and a dryer or both a washer and an iron or both a dryer and an iron. Hence, $B = (W \cap D) \cup (W \cap I) \cup (D \cap I)$. Observing that $(W \cap D) \cap (W \cap I)$, $(W \cap D) \cap (D \cap I)$, $(W \cap I) \cap (D \cap I)$, and $(W \cap D) \cap (W \cap I) \cap (D \cap I)$ all equal $W \cap D \cap I$, we can apply the inclusion–exclusion principle to get

$$\begin{aligned} P(B) &= P\big((W \cap D) \cup (W \cap I) \cup (D \cap I)\big) \\ &= P(W \cap D) + P(W \cap I) + P(D \cap I) - 3P(W \cap D \cap I) + P(W \cap D \cap I) \\ &= 0.15 + 0.13 + 0.09 - 2 \cdot 0.05 = 0.27. \end{aligned}$$

c) Refer to parts (a) and (b). Let C denote the event that the customer purchases exactly one of the items. We note that event C^c occurs if and only if the customer purchases either no items (event A) or two or more items (event B); in other words, $C^c = A \cup B$. Hence, from the complementation rule, the additivity axiom, and the results of parts (a) and (b),

$$P(C) = 1 - P(C^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) = 1 - 0.39 - 0.27 = 0.34.$$

2.102 Let N and L denote the events that, during any given hour, you receive a nonlegitimate e-mail message and a legitimate e-mail message, respectively. From the problem statement, we have $P(N) = 0.5$, $P(L) = 0.7$, and $P(N \cap L) = 0.4$. The event that you receive no e-mail message during a given hour is $N^c \cap L^c$. Hence, from De Morgan's law, the complementation rule, and the general addition rule,

$$\begin{aligned} P\big(N^c \cap L^c\big) &= P\big((N \cup L)^c\big) = 1 - P(N \cup L) = 1 - \big(P(N) + P(L) - P(N \cap L)\big) \\ &= 1 - (0.5 + 0.7 - 0.4) = 0.2. \end{aligned}$$

Theory Exercises

2.103 We want to use mathematical induction to prove that, for all $N \in \mathcal{N}$,

$$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n).$$

Equation (2.4) implies that this result holds for $N = 2$. Assuming its truth for N, we prove it for $N + 1$. So, assume that $A_1, \ldots, A_{N+1}$ are mutually exclusive events. Then $A_1, \ldots, A_N$ are mutually exclusive. Furthermore, from the distributive law for sets,

$$\left(\bigcup_{n=1}^{N} A_n\right) \cap A_{N+1} = \bigcup_{n=1}^{N} (A_n \cap A_{N+1}) = \bigcup_{n=1}^{N} \emptyset = \emptyset,$$

so that $\bigcup_{n=1}^{N} A_n$ and A_{N+1} are mutually exclusive. Therefore, from Equation (2.4) and the induction assumption,

$$\begin{aligned} P\left(\bigcup_{n=1}^{N+1} A_n\right) &= P\left(\left(\bigcup_{n=1}^{N} A_n\right) \cup A_{N+1}\right) = P\left(\bigcup_{n=1}^{N} A_n\right) + P(A_{N+1}) \\ &= \sum_{n=1}^{N} P(A_n) + P(A_{N+1}) = \sum_{n=1}^{N+1} P(A_n), \end{aligned}$$

as required. It is still necessary to assume Equation (2.5) instead of simply Equation (2.4) because we need additivity to hold for a countably infinite number of mutually exclusive events—and that condition does not follow from finite additivity.

2.104

a) If events A and B both have probability 0, then, by the nonnegativity axiom and the general addition rule,

$$0 \le P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B) = 0 + 0 = 0.$$

Hence, $P(A \cup B) = 0$.

b) If events A and B both have probability 1, then, from the complementation rule, A^c and B^c both have probability 0. Therefore, from the complementation rule, De Morgan's law, and part (a),

$$P(A \cap B) = 1 - P\big((A \cap B)^c\big) = 1 - P(A^c \cup B^c) = 1 - 0 = 1.$$

c) We prove that parts (a) and (b) hold for countably many events. For part (a), let $A_1, A_2, \ldots$ be events, all of which have probability 0. Then, by the nonnegativity axiom and Boole's inequality (Exercise 2.75),

$$0 \le P\left(\bigcup_n A_n\right) \le \sum_n P(A_n) = \sum_n 0 = 0.$$

Hence, $P\big(\bigcup_n A_n\big) = 0$. For part (b), let $A_1, A_2, \ldots$ be events, all of which have probability 1. Then, by the complementation rule, $A_1^c, A_2^c, \ldots$ all have probability 0. Therefore, from the complementation rule, De Morgan's law, and the result we just proved,

$$P\left(\bigcap_n A_n\right) = 1 - P\left(\left(\bigcap_n A_n\right)^c\right) = 1 - P\left(\bigcup_n A_n^c\right) = 1 - 0 = 1.$$

d) We will provide counterexamples in both cases. Let $\Omega = [0, 1]$, endowed with a geometric probability model, so that, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{1} = |E|,$$

where E denotes the length of the set E. For each $\omega \in \Omega$, let $A_\omega = \{\omega\}$. Then, $P(A_\omega) = |\{\omega\}| = 0$ for all $\omega \in \Omega$. However,

$$P\left(\bigcup_{\omega \in \Omega} A_\omega\right) = P\left(\bigcup_{\omega \in \Omega} \{\omega\}\right) = P(\Omega) = 1 \ne 0.$$

Hence, we have found an uncountable collection of events with the property that each event has probability 0, but the union of the events does not have probability 0. Now let $B_\omega = A_\omega^c$ for each $\omega \in \Omega$. From the complementation rule, each B_ω has probability 1. However, the intersection of the B_ωs is empty and, hence, has probability 0, not 1.

Advanced Exercises

2.105 In each part, we will provide two solutions, one that uses formal techniques developed in this chapter and the other that applies a slicker approach. Let A denote the event that the second card drawn is an ace. Also, for convenience, we label the cards with the numbers 1–52, where the first four cards are the aces. Let $S = \{1, 2, \ldots, 52\}$.

a) A typical outcome for this random experiment can be represented as an ordered pair, (x, y), where x and y are the numbers of the first and second cards drawn, respectively. Hence, because the first card is replaced before the second card is drawn, a sample space is $\Omega = \{(x, y) : x, y \in S\}$. As the cards are randomly drawn, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{52 \cdot 52} = \frac{N(E)}{2704}.$$

Now,

$$A = \left\{ (x, y) : x \in S,\ \ y \in \{1, 2, 3, 4\} \right\}.$$

Consequently,

$$P(A) = \frac{N(A)}{2704} = \frac{52 \cdot 4}{2704} = \frac{1}{13}.$$

A slicker solution is as follows: Because the first card is replaced in the deck before the second card is drawn, the second draw behaves exactly like the first draw. Hence, the probability that the second card drawn is an ace is the same as the probability that the first card drawn is an ace, which is $4/52 = 1/13$.

b) A typical outcome for this random experiment can be represented as an ordered pair, (x, y), where x and y are the numbers of the first and second cards drawn, respectively. Hence, because the first card is not replaced before the second card is drawn, a sample space is $\Omega = \{ (x, y) : x, y \in S \text{ and } y \neq x \}$. As the cards are randomly drawn, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{52 \cdot 52 - 52} = \frac{N(E)}{52 \cdot 51} = \frac{N(E)}{2652}.$$

Now,

$$A = \left\{ (x, y) : x \in S,\ \ y \in \{1, 2, 3, 4\},\ \text{and } y \neq x \right\}.$$

Consequently,

$$P(A) = \frac{N(A)}{2652} = \frac{52 \cdot 4 - 4}{2652} = \frac{51 \cdot 4}{2652} = \frac{1}{13}.$$

A slicker solution is as follows: By symmetry, the second card drawn is equally likely to be any one of the 52 cards. Hence, the probability that the second card drawn is an ace is $4/52 = 1/13$.

2.106

a) From Exercise 1.62(b), we have $A \setminus B = A \cap B^c$, which is the event that A occurs but B doesn't.

b) From the law of partitions, we get that

$$P(A \setminus B) = P(A \cap B^c) = P(A) - P(A \cap B).$$

c) If $B \subset A$, then $A \cap B = B$. Hence, from part (b),

$$P(A \setminus B) = P(A) - P(A \cap B) = P(A) - P(B).$$

2.107

a) From the solution to Exercise 1.63, we know that $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Consequently, in view of the solution to Exercise 2.106(a), we see that $A \triangle B$ is the event that either A occurs but B doesn't or B occurs but A doesn't. In other words, $A \triangle B$ is the event that exactly one of A and B occurs.

b) We note that $A \setminus B$ and $B \setminus A$ are mutually exclusive events. Hence, from part (a), the additivity axiom, and Exercise 2.106(b), we get

$$\begin{aligned} P(A \triangle B) &= P\big((A \setminus B) \cup (B \setminus A)\big) = P(A \setminus B) + P(B \setminus A) \\ &= \big(P(A) - P(A \cap B)\big) + \big(P(B) - P(A \cap B)\big) \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

2.108

a) We have that $\omega \in \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right)$ if and only if there is an $n \in \mathcal{N}$ such that $\omega \in \bigcap_{k=n}^{\infty} A_k$, that is, if and only if there is an $n \in \mathcal{N}$ such that $\omega \in A_k$ for all $k \geq n$, which is the case if and only if ω is in all but finitely many of the A_ns. Therefore, we see that $\bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right)$ is the event that all but finitely many of the A_ns occur.

b) We have that $\omega \in \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$ if and only if $\omega \in \bigcup_{k=n}^{\infty} A_k$ for each $n \in \mathcal{N}$, that is, if and only if for each $n \in \mathcal{N}$ there is a $k \geq n$ such that $\omega \in A_k$, which is the case if and only if ω is in infinitely many of the A_ns. Therefore, we see that $\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$ is the event that infinitely many of the A_ns occur.
c) If all but finitely many of the A_ns occur, then infinitely many of the A_ns occur. Hence, in view of parts (a) and (b), we have $\bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right) \subset \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$.

2.109 From the domination principle and the certainty axiom, we have $P(E) \leq P(\Omega) = 1$ for all events E; thus, $P(E) \leq 1$ for all events E. For $1 \leq k \leq 8$, let A_k denote the event of a "5" on the kth throw of the die and set $A = \bigcup_{k=1}^{8} A_k$. Your colleague's reasoning shows that $P(A) = 1.333\ldots$, which is impossible because an event must have probability at most 1. Your colleague's reasoning is faulty because he or she used the additivity axiom on the non–mutually exclusive events $A_1, \ldots, A_8$.

2.110 A typical outcome of the random experiment of throwing a die three times can be represented as a triple (x_1, x_2, x_3), where x_k denotes the result of the kth throw. Hence, a sample space is

$$\Omega = \left\{ (x_1, x_2, x_3) : x_1, x_2, x_3 \in \{1, 2, 3, 4, 5, 6\} \right\}.$$

Because the die is balanced, a classical probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6 \cdot 6 \cdot 6} = \frac{N(E)}{216}.$$

For $1 \leq n \leq 3$, let A_n denote the event of a "5" on the nth throw of the die and set $A = \bigcup_{n=1}^{3} A_n$. We want to determine $P(A)$. Now,

$$\begin{aligned} P(A_1) &= P\Big(\left\{ (5, x_2, x_3) : x_2, x_3 \in \{1, 2, 3, 4, 5, 6\} \right\} \Big) \\ &= \frac{N\Big(\left\{ (5, x_2, x_3) : x_2, x_3 \in \{1, 2, 3, 4, 5, 6\} \right\} \Big)}{216} \\ &= \frac{6 \cdot 6}{216} = \frac{1}{6}, \end{aligned}$$

and, likewise, $P(A_2) = P(A_3) = 1/6$. Also,

$$\begin{aligned} P(A_1 \cap A_2) &= P\Big(\left\{ (5, 5, x_3) : x_3 \in \{1, 2, 3, 4, 5, 6\} \right\} \Big) \\ &= \frac{N\Big(\left\{ (5, 5, x_3) : x_3 \in \{1, 2, 3, 4, 5, 6\} \right\} \Big)}{216} \\ &= \frac{6}{216} = \frac{1}{36}, \end{aligned}$$

and, likewise, $P(A_1 \cap A_3) = P(A_2 \cap A_3) = 1/36$. Also,

$$P(A_1 \cap A_2 \cap A_3) = P\big(\{(5, 5, 5)\} \big) = \frac{1}{216}.$$

Hence, from the inclusion–exclusion principle,

$$\begin{aligned} P(A) &= P\left(\bigcup_{n=1}^{3} A_n\right) \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\ &\quad + P(A_1 \cap A_2 \cap A_3) \\ &= 3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{36} + \frac{1}{216} = \frac{91}{216} \approx 0.421. \end{aligned}$$

2.111

a) On any particular toss of the die, we are interested only in whether the result is a six or not, which we denote s and f, respectively. The game stops at the nth toss of the die if and only if the first $n-1$ tosses are not sixes and the nth toss is a six, which we can represent as the n-tuple, $(f, f, \ldots, f, s)$. Hence, a sample space is

$$\Omega = \Big\{(\underbrace{f, f, \ldots, f}_{n-1 \text{ times}}, s) : n \in \mathcal{N}\Big\} \cup \{(f, f, \ldots)\},$$

where $(f, f, \ldots)$ represents the outcome that a six is never tossed (i.e., the game never stops).

b) Because the die is balanced, when it is tossed n times, the possible outcomes are equally likely. From the hint, there are 6^n possible outcomes of which 5^{n-1} have the property that the first $n-1$ tosses are not sixes and the nth toss is a six. Hence,

$$P\big(\{(\underbrace{f, f, \ldots, f}_{n-1 \text{ times}}, s)\}\big) = \frac{5^{n-1}}{6^n} = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}.$$

As we discover in part (c), the game must eventually stop. This implies that $P\big(\{(f, f, \ldots)\}\big) = 0$.

c) Let E denote the event that the game eventually stops. Then, from Proposition 2.2 on page 42 and part (b), we have

$$P(E) = \sum_{n=1}^{\infty} P\big(\{(\underbrace{f, f, \ldots, f}_{n-1 \text{ times}}, s)\}\big) = \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n-1} = \frac{1}{6} \cdot \frac{1}{1 - 5/6} = 1.$$

d) Let T denote the event that Tom wins. As Tom goes first, event T occurs if and only if the first six occurs on trial 1 or 4 or 7 or ..., that is, on trial $3n-2$ for some $n \in \mathcal{N}$. Hence, from Proposition 2.2 and part (b), we have

$$\begin{aligned} P(T) &= \sum_{n=1}^{\infty} P\big(\{(\underbrace{f, f, \ldots, f}_{3n-3 \text{ times}}, s)\}\big) = \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{3n-3} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\left(\frac{5}{6}\right)^{3}\right)^{n-1} \\ &= \frac{1}{6} \cdot \frac{1}{1 - (5/6)^3} = \frac{36}{91} \approx 0.396. \end{aligned}$$

Let D denote the event that Dick wins. As Dick goes second, event D occurs if and only if the first six occurs on trial 2 or 5 or 8 or ..., that is, on trial $3n-1$ for some $n \in \mathcal{N}$. Hence, from Proposition 2.2

and part (b), we have

$$P(D) = \sum_{n=1}^{\infty} P\big(\{(\underbrace{f, f, \ldots, f}_{3n-2 \text{ times}}, s)\}\big) = \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{3n-2} = \frac{1}{6} \cdot \frac{5}{6} \sum_{n=1}^{\infty} \left(\left(\frac{5}{6}\right)^{3}\right)^{n-1}$$

$$= \frac{5}{36} \cdot \frac{1}{1-(5/6)^3} = \frac{30}{91} \approx 0.330.$$

Let H denote the event that Harry wins. Then, from part (c), the complementation rule, and the additivity axiom, we deduce that

$$P(H) = 1 - P(H^c) = 1 - P(T \cup D) = 1 - P(T) - P(D) = 1 - \frac{36}{91} - \frac{30}{91} = \frac{25}{91} \approx 0.275.$$